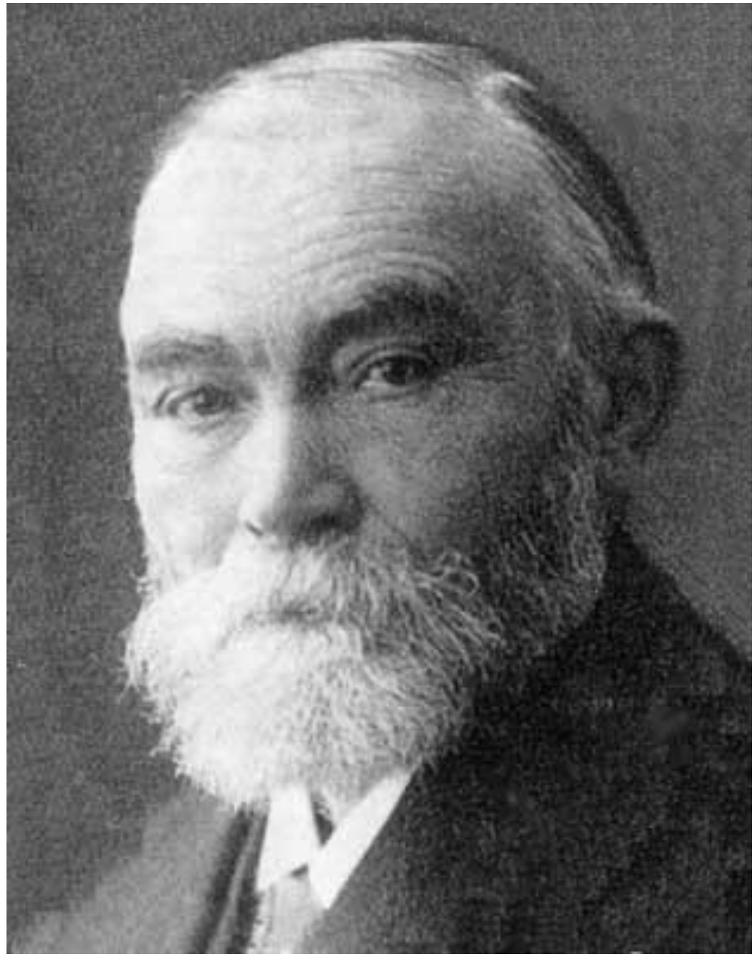
Jean Van Heijenoort's View of Modern Logic

Jean van Heijenoort (1912-1986)



"Modern logic began in 1879, the year in which Gottlob Frege (1848–1925) published his *Begriffsschrift.*"

p. 242, Jean van Heijenoort, "Historical Development of Modern Logic", *Modern Logic* **2**, 242–255.
[Prepared by Irving H. Anellis from a previously unpublished typescript of 1974.]

“In less than ninety pages this booklet presented a number of discoveries that changed the face of logic. The central achievement of the work is the theory of quantification; but this could not be obtained till the traditional decomposition of the proposition into subject and predicate had been replaced by its analysis into function and argument(s). A preliminary accomplishment was the propositional calculus, with a truth-functional definition of the connectives, including the conditional. Of cardinal importance was the realization that, if circularity is to be avoided, logical derivations are to be *formal,* that is, have to proceed according to rules that are devoid of any intuitive logical force but simply refer to the typographical form of the expressions; thus the notion of formal system made its appearance. The rules of quantification theory, as we know them today, were then introduced. The last part of the book belongs to the *foundations of mathematics,* rather than to logic, and presents a logical definition of the notion of mathematical sequence. Frege’s contribution marks one of the sharpest breaks that ever occurred in the development of a science.”

p. 242, Jean van Heijenoort, “Historical Development of Modern Logic”, *Modern Logic* **2**, 242–255. [Prepared by Irving H. Anellis from a previously unpublished typescript of 1974.]

The major characteristics of modern mathematical logic, as found in the *Begriffsschrift* and first introduced by Frege:

1. a propositional calculus with a truth-functional definition of connectives, especially the conditional;

2. decomposition of propositions into function and argument instead of into subject and predicate;

3. a quantification theory, based on a system of axioms and inference rules; and

4. definitions of *infinite sequence* and *natural number* in terms of logical notions (*i.e*. the logicization of mathematics).

Jean van Heijenoort, “Logic as Calculus and Logic as Language”, *Synthèse* **17**, 324–330; reprinted in Robert S. Cohen & Marx W. Wartofsky (eds.), *Boston Studies in the Philosophy of Science 3 (1967), In Memory of Russell Norwood Hansen, Proceedings of the Boston Colloquium for Philosophy of Science 1964/1965* (Dordrecht: D. Reidel, 1967), 440–446.

In addition, Frege, according to van Heijenoort (and adherents of the historio-graphical conception of a “Fregean revolution”):

5. presented and clarified the concept of *formal system*;

and

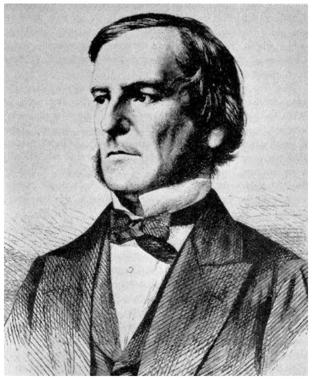
6. made possible, and gave, a use of logic for philosophical investigations (especially for philosophy of language).

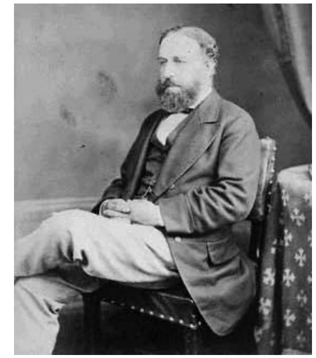
Jean van Heijenoort, “Logic as Calculus and Logic as Language”, *Synthèse* **17**, 324–330; reprinted in Robert S. Cohen & Marx W. Wartofsky (eds.), *Boston Studies in the Philosophy of Science 3 (1967), In Memory of Russell Norwood Hansen, Proceedings of the Boston Colloquium for Philosophy of Science 1964/1965* (Dordrecht: D. Reidel, 1967), 440–446.

In “On the Frege-Russell Definition of Number” (undated ms.), van Heijenoort added:

7. distinguishing singular propositions, such as “Socrates is mortal” from universal propositions such as “All Greeks are mortal.”

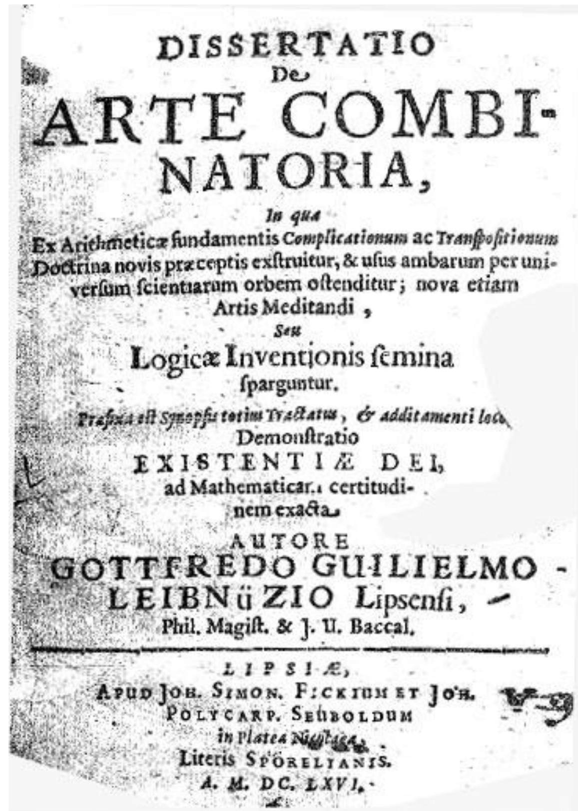
Logic as Calculus





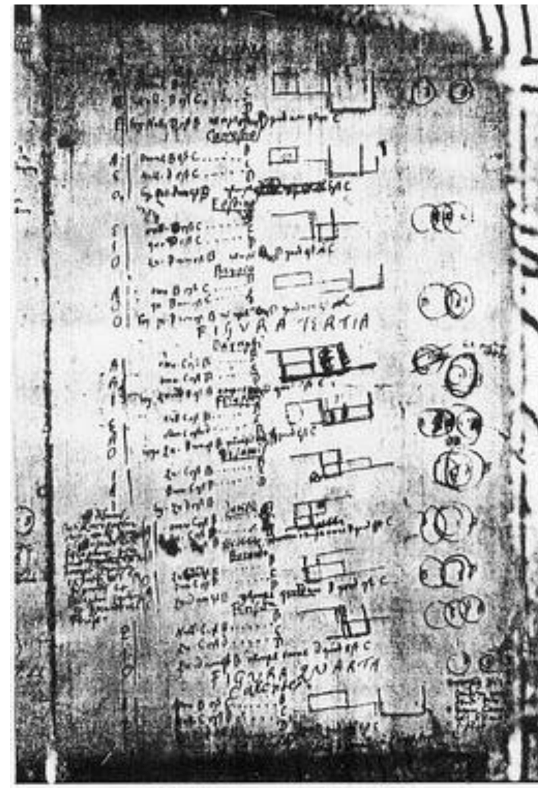
"Considered by itself, the period would, no doubt, leave its mark on the history of logic, but it would not count as a great epoch." – *From Frege to Gödel*, p. vi

Logic as Language



DISSERTATIO
DE
ARTE COMBI-
NATORIA,
In qua
Ex Arithmeticæ fundamentis Complicationum ac Transpositionum
Doctrina novis præceptis exstruitur, & usus ambarum per uni-
versum scientiarum orbem ostenditur; nova etiam
Artis Meditandi,
Seu
Logicæ Inventionis semina
sparguntur.
Præfixa est Synopsis totius Tractatus, & additamenti loco
Demonstratio
EXISTENTIÆ DEI,
ad Mathematicar.ı certitudi-
nem exacta.
AUTORE
GOTTFREDO GUILIELMO
LEIBNüZIO Lipsensi,
Phil. Magist. & J. U. Baccal.

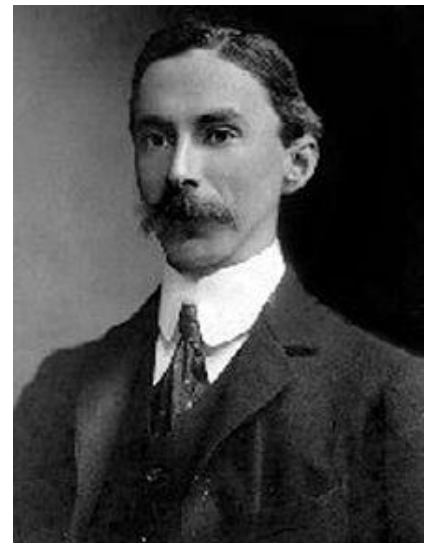
LIPSIÆ,
APUD JOH. SIMON. FICKIUM ET JOH.
POLYCARP. SEUBOLDUM
in Platea Nicolæa.
Literis SPÖRELIANIS.
A. M. DC. LXVI.



Syllogistic diagrams by Leibniz (36.17)

“heralded by Leibniz” – *From Frege to Gödel,* p. vi

Logic as Language



"Frege's work was slow in winning recognition. …Frege's *Begriffsschrift* and Peano's *Arithmeticies principia*…led to *Principia mathematica*." – *From Frege to Gödel*, p. vi

Logic as Language	/ **Logic as Calculus**
Absolutism	Relativism
Universality (Universal universe of discourse (*Universum*) fixed, with nothing extra-systematic, to logically reconstruct the universe	Universes of discourse, with multiple interpretations
Formal deductive system	“ignore proofs”
Syntactic and semantic * Frege: Wertverlauf semantic; * Russell: set-theoretic semantic)	Syntactic (model-theoretic): individuals and relations
Function-theoretic structure of props	Algebraic structure of props “tried to copy mathematics too closely, often artificially” (– *From Frege to Gödel*, p. vi)

“Hilbert’s position is somewhat between that of Frege-Russell and that of Peirce-Schröder-Löwenheim…”

-- p. 185, Jean van Heijenoort, “Set-theoretic Semantics”, in Robin Oliver Gandy & John Martin Elliott Holland (eds.), *Logic Colloquium ’76, (Oxford, 1976)* (North-Holland, Amsterdam, 1977), 183–190

$$\sum_{1}^{n}\lambda\; a_{\lambda} = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$\prod_{1}^{n}\lambda\; a_{\lambda} = a_1 a_2 a_3 \ldots a_{n-1} a_n$$

Schröder, *Vorlesungen uber die Algebra der Logik (Exakte Logik)*, Bd. II, 1891, § 30, 35

"Löwenheim's paper (1915), which links up with the work of Schröder, brings to the fore notions (validity, decision methods)…." – *From Frege to Gödel*, p. vii

Including: Löwenheim-Skolem Theorem, Herbrand's Fundamental Theorem

N° D'ORDRE 2121

SÉRIE A, N° DE SÉRIE 1252

THÈSES

PRÉSENTÉES

A LA FACULTÉ DES SCIENCES DE PARIS

POUR OBTENIR

LE GRADE DE DOCTEUR ÈS SCIENCES MATHÉMATIQUES

PAR

M. Jacques HERBRAND

1re THÈSE

Recherches sur la théorie de la démonstration

2me THÈSE

Propositions données par la faculté

---

11 Juin

Soutenues le 1930 devant la commission d'examen

Président : M. VESSIOT

Examinateurs { MM. DENJOY, FRECHET

Jedes Mathem. Problem kann auf folgende Frage zurückgeführt werden:

Gegeben ist ein Gesetz, vermöge ~~[illegible]~~ dessen aus ~~aus~~ jeder vorgelegten nur aus Null und 1 ~~[illegible]~~ bestehenden ~~[illegible]~~ nichtabbrechenden Reihe

(α) 0 0 1 1 0 0 . . . .

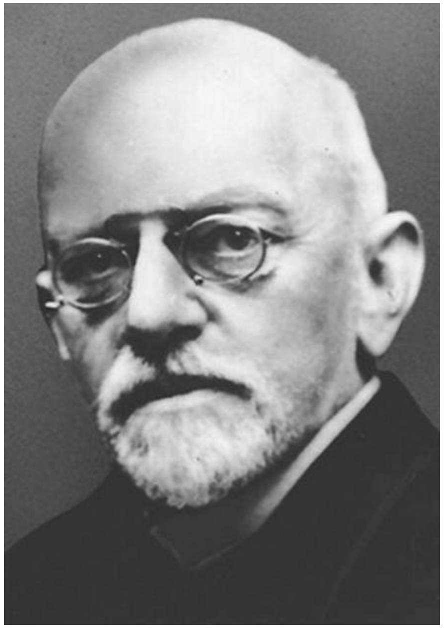
eine bestimmte andere solche Reihe (durch Reihenoperationen: grösste Ganze suchen, Teiler suchen etc. nicht Würfeln..)

(β) 1 0 1 1 1 . . —

construiert werden kann. ~~[illegible]~~ Man soll durch eine endliche Zahl von Operationen entscheiden, ~~[illegible]~~ ob in irgend einer Reihe (β) eine 0 vorkommt oder ob alle Reihen (β) nur Einsen bestehen?

Ich behaupte: Jede solche Entscheidung ist durch eine endliche Zahl von Operationen (Reihenoperationen) möglich. d.h. Es giebt kein Gesetz, bei welchem die Entscheidung nicht durch eine endliche Zahl von Operationen möglich wäre. d.h. Jedes math. Problem ist lösbar. Alles dem menschlichen Verstande erreichbare (durch reines Denken ohne Materie) ist auch aufzulösen. Es giebt nur ein Problem. (z.B. Quadratur des Kreises. Hat $\pi = 3{,}14..$ 10 aufeinander folgende 7 en etc.)

Von der Annahme der Möglichkeit geht man von vorneherein aus.



Transcription:
Jedes Mathem. Problem kann auf folgende Frage zurückgeführt werden: Gegeben ist ein Gesetz, vermöge dessen zu jeder vorlegten ‹vorgelegten› nur aus Null und 1 bestehenden nicht abbrechenden Reihe
(α) 0 0 1 1 0 0 ...
eine bestimmte andere solche Reihe (durch Rechenoperationen: Grösste Ganze suchen, Teilersuche etc. nicht Würfeln...)
(β) 1 0 0 1 1 1....
construiert ‹konstruiert› werden kann. Man soll durch eine endliche Zahl von Operationen entscheiden ob in irgendeiner Reihe (β) eine 0 vorkommt oder ob alle Reihen (β) nur ‹aus› Einsen bestehen?
Ich behaupte: Jede solche Entscheidung ist durch eine endliche Zahl von Operationen (Rechenoperationen) möglich«,» d.h. es giebt kein Gesetz, bei welchem die Entscheidung nicht durch eine endliche Zahl von Operationen möglich wäre. D.h. jedes math. Problem ist lösbar. Alles dem menschlichen Verstande erreichbare (durch reines Denken ohne Materie) ist auch aufzulösen. Es giebt nur ein Problem. (z.B. Quadratur des Kreises, hat π = 3,14... 10 aufeinander folgende 7en, etc.) Von der Annahme der Möglichkeit geht man von vorne herein ‹vornherein› aus.
Translation
Every mathematical problem can be reduced to the following question: A rule is given, by which to every given non-breaking off sequence consisting only of zero and 1
(α) 0 0 1 1 0 0...
a certain other such sequence (by computational operations: «Grösste Ganze Suchen», etc. not throwing dice...)
(β) 1 0 0 1 1 1....
can be constructed. One is to decide by a finite number of operations whether if zero occurs in some sequence (β) or all sequences (β) consist of only ones.
I claim: every such decision is possible through a finite number of operations (computational operations), i.e. there is no rule, with which the decision would not be possible by a finite number of operations, i.e. every math. problem is solvable. All that the human intellect can reach (by pure thinking without matters) is also to be dissolved. There is only one problem (e.g. quadrature of the circle, π = 3. 14 has ... 10 successive 7's etc.) One proceeds at the outset from the assumption of the possibility.

“Gödel’s Theorem”, Paul Edwards (ed.), *Encyclopedia of Philosophy*, vol. 3 (New York: Macmillan, 1963), 348–357;
“Logical Paradoxes”, Paul Edwards (ed.), *Encyclopedia of Philosophy*, vol. 5 (New York: Macmillan, 1963), 45–51;
“Système et métasystème chez Russell”, in The Paris Logic Group (eds.), *Logic Colloquium ’85* (North-Holland, Amsterdam/London/New York, 1987), 111–122;
“Nature of Logic” (undated manuscript notes):

The consequence of the universality of Frege’s and Russell’s conception of logic is that, although issues regarding the properties of their logical systems arise, in particular the properties of consistency and completeness, they have to rely upon *ad hoc* extra-logical devices in their attempts to deal with these issues; they cannot deal with these **within** their logical systems.

Ex. 1. Russell’s 1908 theory of types to deal with the Russell set, requires introduction of non-logical or non-intuitive, or at least debatable axioms:

- Multiplicative Axiom (Axiom of Choice)
- Axiom of Infinity

Bertrand Russell, “Mathematical Logic as Based on the Theory of Types”, *American Journal of Mathematics* 30 (1908), 222–262

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I[1]).

Von **Kurt Gödel** in Wien.

1.

Die Entwicklung der Mathematik in der Richtung zu größerer Exaktheit hat bekanntlich dazu geführt, daß weite Gebiete von ihr formalisiert wurden, in der Art, daß das Beweisen nach einigen wenigen mechanischen Regeln vollzogen werden kann. Die umfassendsten derzeit aufgestellten formalen Systeme sind das System der Principia Mathematica (PM)[2]) einerseits, das Zermelo-Fraenkelsche (von J. v. Neumann weiter ausgebildete) Axiomensystem der Mengenlehre[3]) andererseits. Diese beiden Systeme sind so weit, daß alle heute in der Mathematik angewendeten Beweismethoden in ihnen formalisiert, d. h. auf einige wenige Axiome und Schlußregeln zurückgeführt sind. Es liegt daher die Vermutung nahe, daß diese Axiome und Schlußregeln dazu ausreichen, alle mathematischen Fragen, die sich in den betreffenden Systemen überhaupt formal ausdrücken lassen, auch zu entscheiden. Im folgenden wird gezeigt, daß dies nicht der Fall ist, sondern daß es in den beiden angeführten Systemen sogar relativ einfache Probleme aus der Theorie der gewöhnlichen ganzen Zahlen gibt[4]), die sich aus den Axiomen nicht

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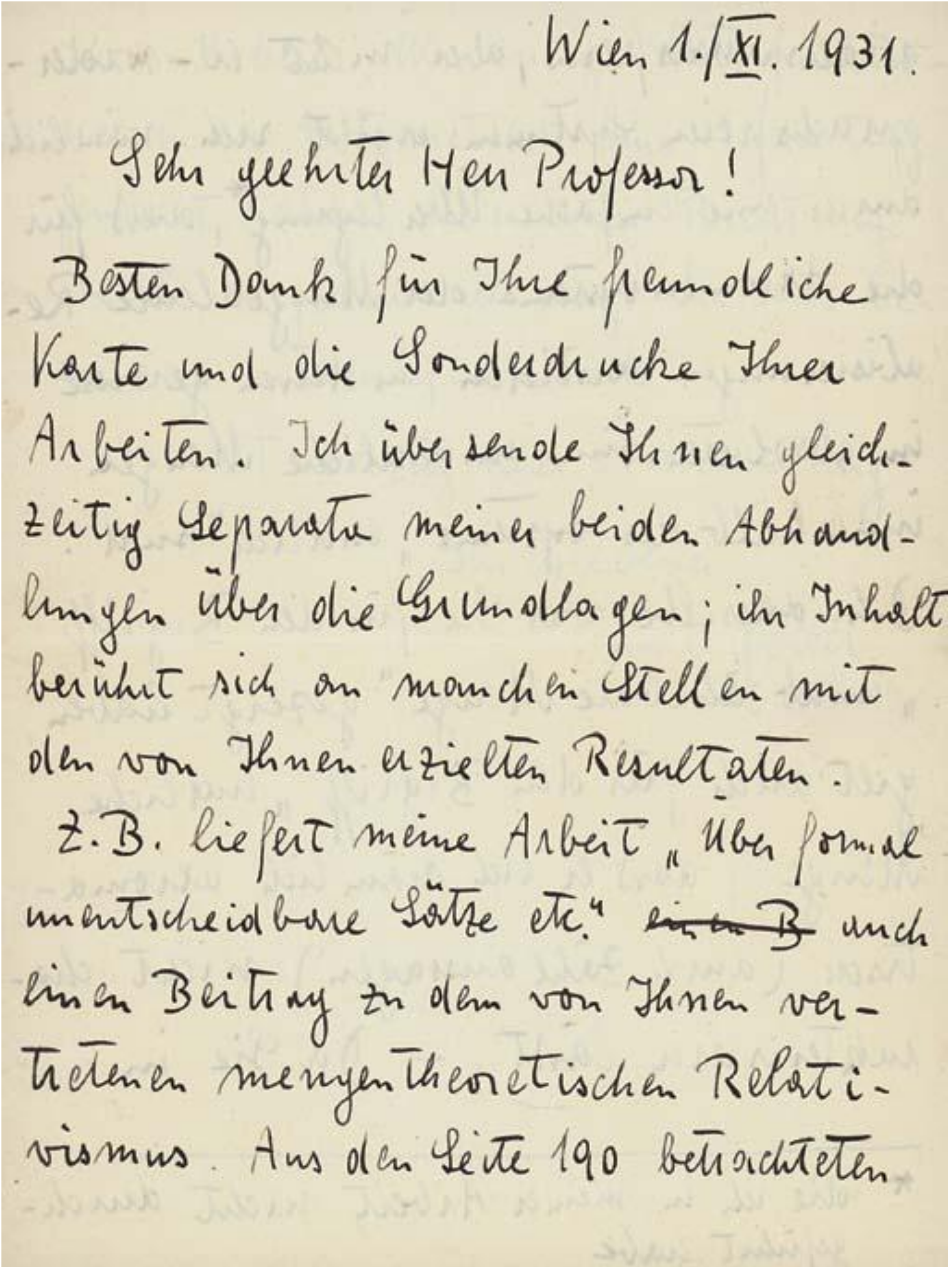
[1]) Vgl. die im Anzeiger der Akad. d. Wiss. in Wien (math.-naturw. Kl.) 1930, Nr. 19 erschienene Zusammenfassung der Resultate dieser Arbeit.

[2]) A. Whitehead und B. Russell, Principia Mathematica, 4. Aufl., Cambridge 1929. Zu den Axiomen des Systems PM rechnen wir insbesondere auch: Das Unendlichkeitsaxiom (in der Form: es gibt genau abzählbar viele Individuen), das Reduzibilitäts- und das Auswahlaxiom (für alle Typen).

[3]) Vgl. A. Fraenkel, Zehn Vorlesungen über die Grundlegung der Mengenlehre, Wissensch. u. Hyp. Bd. XXXI. J. v. Neumann, Die Axiomatisierung der Mengenlehre. Math. Zeitschr. 22, 1925. Journ. f. reine u. angew. Math. 155 (1929), 169 (1929). Wir bemerken, daß man zu den in der angeführten Literatur gegebenen mengentheoretischen Axiomen noch die Axiome und Schlußregeln des Logikkalküls hinzufügen muß, um die Formalisierung zu vollenden. — Die nachfolgenden Überlegungen gelten auch für die in den letzten Jahren von D. Hilbert und seinen Mitarbeitern aufgestellten formalen Systeme (soweit diese bisher vorliegen). Vgl. D. Hilbert, Math. Ann. 81, Abh. aus d. math. Sem. der Univ. Hamburg I (1920), VI (1925). P. Bernays, Math. Ann. 93. J. v. Neumann, Math. Zeitschr. 27 (1925). W. Ackermann, Math. Ann. 90.

[4]) D. h. genauer, es gibt unentscheidbare Sätze, in denen außer den logischen Konstanten: $\overline{}$ (nicht), $\vee$ (oder), (x) (für alle), $=$ (identisch mit) keine anderen Begriffe vorkommen als $+$ (Addition), . (Multiplikation), beide bezogen auf natürliche Zahlen, wobei auch die Präfixe (x) sich nur auf natürliche Zahlen beziehen dürfen.


Wien 1./XI. 1931.

Sehr geehrter Herr Professor!

Besten Dank für Ihre freundliche Karte und die Sonderdrucke Ihrer Arbeiten. Ich übersende Ihnen gleichzeitig Separata meiner beiden Abhandlungen über die Grundlagen; ihr Inhalt berührt sich an manchen Stellen mit den von Ihnen erzielten Resultaten.

Z.B. liefert meine Arbeit „Über formal unentscheidbare Sätze etc." ~~ein B~~ auch einen Beitrag zu dem von Ihnen vertretenen mengentheoretischen Relativismus. Aus den Seite 190 betrachteten

Thank you very much for your friendly card and the offprints of your papers. I am concurrently sending you reprints of my two essays regarding the fundamentals; several passages therein relate to the results that you obtained. For example, my paper entitled "Über formal unentscheidbare Sätze etc." also provides a contribution to the set-theoretical relativism held by you. This is because, as shown by a simple calculation*, the consistent, but not ω-consistent systems examined on page 190 indicate that there exist realizations for axiom systems in set theory in which certain quantities that are infinite from an absolute standpoint are "finite" within the system. In other words, that which you showed for the term "uncountable quantity" also holds true for the term "finite quantity," namely that it cannot be axiomatically characterized (expressed by a number). Since you made a suggestion in your paper "Über einige Satzfunktionen in der Arithmetik" which points in this direction, I think you will find this particularly interesting.

Ex. 2: *Gödel incompleletenss*: **not merely** the formal undecidability of some theorems of number theory (*Ex*: Fermat's Last Theorem; Goldbach's Conjecture; Riemann's Hypothesis), of both *T* and ~*T within the formal system*

BUT also:

The unprovability within any formal axiomatic system adequate for number theory of the consistency of that system; i.e.

Satz XI: *Sei* κ *eine beliebige rekursive widerspruchsfreie Klasse von* FORMELN, *dann gilt: Die* SATZFORMEL*, welche besagt, daß* κ *widerspruchsfrei ist, ist nich* κ-BEWEISBAR; inbesondere ist die Widerspruchsfreiheit von *P* in *P* unbeweis-bar, vorausgetzt, daß *P* widerspruchsfrei ist (im entgegengesetzten Fall ist natürlich jede Aussage beweisbar).

Thus: for any consistent formal system Z (of PRA) adequate for axiomatizing the sequence of natural numbers, the consistency of Z is unprovable in Z:

$$\nvdash_{Z} \mathrm{Con}(Z)$$

Precisely van Heijenoort’s point w.r.t. the incompleteness theorems, and in particular to the second incompleteness theorem,

- in his *Encyclopedia of Philosophy* article “Gödel’s Theorem” indirectly;
- in his “Nature of Logic” notes tacitly; and
- in his seminar lectures on “Foundation of Mathematics” very explicitly,

was that the **complex, multi-layered** proof by Gödel, especially w.r.t. the second incompleteness theorem, is that the constant shifting back-and-forth between the system and metasystem, the syntactic and the semantic levels, and between *true* and *provable*, is **necessitated**

for universal systems, i.e. systems in which there is nothing extra-systematic – *Principia*-like – systems

because the proofs concerning properties of those systems **CANNOT** be carried within those systems.

The sets of manuscript notes “Nature of Logic” comprise van Heijenoort’s attempts to understand:

1) the relation between logic and metalogic (which he was calling in these notes “basic logic”); and
2) the relation of logic (formal or ideal language) to Ordinary Language,

where the basic logic is the tool for investigating and comparing the properties of various logical systems – and to do so without falling into a Carnapian metalinguistic regress.

In “Nature of Logic”, the issue reaches no resolution.

Summary

- Aristotle: traditional (ancient) logic (van Heijenoort barely considered this)
- Leibniz: the conception of modern logic
- “Boolean algebra”: the incubation of modern logic
- Frege-Russell: the “birth” of modern logic
- Löwenheim, Skolem, Hilbert, Herbrand: the “schooling” of Frege-Russell
- Gödel: the “graduation” of modern logic to adulthood
- “*après* Gödel” (Gentzen, Hilbert, Herbrand, Beth, Hintikka, Smullyan): the early “adulthood” of modern logic development of “quantification theory”, i.e. proof procedures for $FOL^=$ (considered by van Heijenoort primarily in unpublished manuscripts and in *El desarrollo de la teoría de la cuantificación*; (Turing, Church, Kleene, *et al*.): decidability theory (considered by van Heijenoort primarily in unpublished manuscripts and class lectures, e.g. in “Foundation of Mathematics” seminar)

“Mathematical logic is what logic , through twenty-five centuries and a few transformations, has become today.” – *From Frege to Gödel*, p. vii

From Frege to Gödel intended to document the growth of modern logic from birth with Frege to entry to maturity with Gödel

Spitzfindigkeit [Subtlety]

The word is borrowed from a title of Kant, *Die falsche Spitzfindigkeit der vier syllogistischen Figuren erweisen*, Königsberg 1762. Spitzfindigkeit is subtlety in questions of logical form and divides thinkers into two well-marked camps: on the one side, those who have it: Aristotle, the Stoics, Abelard, Albertus Magnus, Peter of Spain, the logicians of the 14*th* and 15*th* centuries, Leibniz, Frege, Boole, De Morgan; on the other, those who do not have it: Sextus Empiricus, Cicero, the logicians of the 16*th* century, Descartes, the authors of the Port-Royal logic, Kant, Prantl.

The article is a sketch, in black and white, of the history of logic and retells the history, familiar today after the work published in the last ten years, of four great epochs of logic: aristotelian, stoic-megaran, mediævel, and modern. Numerous quotes from Prantl of stunning ignorance show how recent and deep the renewal of history of logic has been.

The article concludes in a militant tone that logic will either be subtle or it will not exist. It will certainly help to disturb the prejudices which still prevail among those who speak of the history of logic without knowing it close up.

English translation by Thomas Drucker of Jean van Heijenoort, Review of I. M. Bocheński, "Spitzfindigkeit", JSL **22**(1957), 382