Abelian Integrals and Categoricity

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April 2, 2012

Abelian integrals

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where $w \in \mathrm{acl}(\mathbb{C}(z))$.

i.e.

$$\int \omega$$

where ω is a meromorphic differential form on a Riemann surface C.

e.g.

$$\int \frac{dz}{\sqrt{z^3 + az + b}} = \int \frac{dz}{w}$$

on $E := \{w^2 = z^3 + az + b\}$.

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Abelian integrals

Multifunction $I_\omega : C(\mathbb{C})^2 \to \mathbb{C}$

$$I_\omega(P, Q) = \int_P^Q \omega,$$

value depends on path from P to Q on C.

Status of $\mathbb{C}_{\int \omega} := < \mathbb{C}; +, \cdot, I_\omega >$?

Example: $\omega = \frac{dz}{z}$ on $C = \mathbb{P}^1$

- $\int_1^b \frac{dz}{z} = \exp^{-1}(b) = \ln(b) + 2\pi i\mathbb{Z}$
- $\int_a^b \frac{dz}{z} = \exp^{-1}(b) - \exp^{-1}(a) = \ln(b) - \ln(a) + 2\pi i\mathbb{Z}$
- So $\mathbb{C}_{\int \omega}$ interdefinable with $\mathbb{C}_{\exp} = < \mathbb{C}; +, \cdot, \exp >$.
- Zilber: conjectural categorical description of $\mathbb{C}_{\exp}$.
- Involves transcendence conjectures - e.g. $e^e \in \mathbb{Q}$?

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Abelian integrals of the first kind

Suppose $\omega \in \Omega :=$ space of *holomorphic* differential forms on a Riemann surface C.
Say $\omega = \omega_1, \ldots, \omega_g$ basis for Ω, where $g = \text{genus}(C)$.
Fix $P_0 \in C(\mathbb{C})$.

Fact (Abel, Jacobi)

C embeds in its Jacobian $J = \text{Pic}^0(C)$ such that

$$\left(\int_{P_0}^{Q} \omega_1, \ldots, \int_{P_0}^{Q} \omega_g\right) = \pi^{-1}(Q)$$

where $\pi : \mathbb{C}^g \twoheadrightarrow J(\mathbb{C})$ is a homomorphism with kernel a lattice (the periods).

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Linear reduct

Let $\mathcal{O} := \{\eta \in \mathrm{Mat}_g(\mathbb{C}) \mid \eta(\ker(\pi)) \leq \ker(\pi)\} \cong \mathrm{End}(J)$.
Let $\mathcal{O}^0 := \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}$.
Consider $\mathbb{C}^g$ as a new sort with just the $\mathcal{O}^0$-module structure:

$$\mathrm{Cov}(J) := \left\langle \begin{array}{c} \langle \mathbb{C}^g; +, (\eta)_{\eta \in \mathcal{O}^0} \rangle \\ \pi : \mathbb{C}^g \to J(\mathbb{C}) \\ \langle \mathbb{C}; +, \cdot \rangle \end{array} \right\rangle$$

Lemma

$T_J := \mathrm{Th}(\mathrm{Cov}(J))$ *has quantifier elimination and axiomatisation:*

$\langle \mathbb{C}^g +, (\eta)_{\eta \in \mathcal{O}^0} \rangle$ *is a* $\mathcal{O}^0$*-module;*
π *is a surjective* $\mathcal{O}$*-homomorphism;*
$\langle \mathbb{C}; +, \cdot \rangle \models \mathrm{ACF}_0$ *.*

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Categoricity

Theorem (Categoricity over $\ker(\pi)$)

Suppose J is defined over a number field. Then $\mathrm{Cov}(J)$ is specified up to isomorphism by:

its first order theory T_J;

its cardinality;

the isomorphism type of $\ker(\pi)$.

- Zilber: analogous statement for $\mathbb{G}_m$.
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Atomicity

$B \subseteq \mathbb{C}$ finite algebraically independent

$$\mathcal{M}_B := \pi^{-1}(J(\mathrm{acl}(B))) \preceq \mathrm{Cov}(J).$$

Lemma (Atomicity)

$\mathcal{M}_B$ is atomic, hence unique, over $\bigcup_{B' \subsetneq B} \mathcal{M}_{B'}$.

Categoricity theorem follows: $\mathrm{Cov}(J)$ is built uniquely over a transcendence basis of $\mathbb{C}$.

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Proof

Equivalent by QE:

Lemma (Atomicity)

- *$\overline{a} \in J(\mathrm{acl}(B))$;*
- *a_i in simple subgroups, no $\mathcal{O}$-linear relations;*
- *$k_\partial := \bigcup_{B' \subsetneq B} \mathrm{acl}(B')$;*

Then exist only finitely many types

$$\mathrm{tp}^{\mathrm{ACF}}((\overline{a}_n)_n / k_\partial).$$

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Proof.

- $k := k_\partial(\bar{a})$

Step I ("Mordell-Weil"): Bound n such that $\bar{a}_n \in J(k)$;

Proof

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Step I ("Mordell-Weil"): Bound n such that $\overline{a}_n \in J(k)$;

Step II ("Kummer"): More generally, bound k-rational imaginaries $\overline{a}_n + Z_n$ for subgroups $Z_n \leq \mathrm{Tor}_n(J)$ - i.e. bound index $[\mathrm{Tor}_n(J) : Z_n]$.

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Step IIa Find number field k_0 such that J and all Z_n may be taken over k_0;

Step IIb By Faltings, the isogenous quotients ${}^{J}/_{Z_n}$ fall into finitely many isomorphism classes; hence reduce to bounding rational points in J as in Step I.

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Step I ("Mordell-Weil"): Bound n such that $\bar{a}_n \in J(k)$; and moreover such that $\eta\bar{a}_n \in J(k)$ for $\eta \in \mathcal{O}$.

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Questions

- Can we relax the assumption that J is over a number field?
- Can we handle semiabelian varieties, and arbitrary abelian integrals?
- Intermediate reducts - status of
  - Complex field + $Q \mapsto \int_{P_0}^{Q} \omega$ for a single ω as a map to a group?
  - Complex field + set Ω + pairing $\int_{P_0}^{\cdot} \cdot : C(\mathbb{C}) \times \Omega \rightarrow < \mathbb{C}; + >$?
  - Algebra structure on the integrals - e.g. two-field exponentiation $< \mathbb{C}; +, \cdot > \twoheadrightarrow^{\text{exp}} < \mathbb{C}; +, \cdot >$? $(\text{trd}(log 2, log 3) = 2?)$
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