Weak Truth Table Degrees of Structures

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Preliminaries

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Recall:

Definition

1. A set $X \subseteq \mathbb{N}$ is Turing reducible to a second set $Y \subseteq \mathbb{N}$ if there is an algorithm that can use Y to decide membership in X.
2. The Turing degree $\deg_T(X)$ of a set X is the class of all subsets of $\mathbb{N}$ that are mutually Turing reducible with X.
3. A set X is weak truth table reducible to a second set Y if there is an algorithm that can use a computably-bounded piece of Y to decide membership in X.
4. The weak truth table degree $\deg_{wtt}(X)$ of a set X is defined in the analogous way.

Preliminaries

Definition

1. A structure is a first-order structure, with universe $\mathbb{N}$, on a finite or countable alphabet $(R_0, R_1, R_2, \ldots)$ of relations. The arities of R_k are computable as a function of k. We identify a structure $\mathcal{A}$ with its atomic diagram

$$D(\mathcal{A}) = \{\langle k, a_1, a_2, \ldots, a_n\rangle : \mathcal{A} \models R_k(a_1, \ldots, a_n)\}.$$

   Note that this is a subset of $\mathbb{N}$.
2. The Turing degree of $\mathcal{A}$, written $\deg_T(\mathcal{A})$, is the Turing degree of $D(\mathcal{A})$.
3. The wtt degree of $\mathcal{A}$ is defined similarly.

Preliminaries

We defined $\deg_T(\mathcal{A})$ as the Turing degree of the atomic diagram of $\mathcal{A}$. Typically, there is a second structure $\mathcal{B}$, isomorphic to $\mathcal{A}$, such that $\deg_T(\mathcal{B}) \neq \deg_T(\mathcal{A})$.

Definition

1. The Turing degree spectrum of $\mathcal{A}$ is the family of all Turing degrees of isomorphic copies of $\mathcal{A}$.

$$\mathrm{spec}_T(\mathcal{A}) = \{\deg_T(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

2. The wtt degree spectrum of $\mathcal{A}$ is

$$\mathrm{spec}_{wtt}(\mathcal{A}) = \{\deg_{wtt}(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

Some motivating examples from the Turing case

Theorem (Knight 86)

If $\mathrm{spec}_T(\mathcal{A})$ *is contained in a countable union* $\bigcup_n C_n$ *of upward cones, then* $\mathrm{spec}_T(\mathcal{A})$ *is contained in a particular* C_{n_0}*.*

Theorem (Hirschfeldt–Khoussainov–Shore–Slinko 02)

If $\mathcal{A}$ *is a nontrivial structure, then there exists a graph* $\mathcal{G}$ *with universe* $\mathbb{N}$ *such that* $\mathrm{spec}_T(\mathcal{G}) = \mathrm{spec}_T(\mathcal{A})$*.*

Theorem (Knight 86)

1. $\mathrm{spec}_T(\mathcal{A})$ *is a singleton if and only if* $\mathcal{A}$ *is trivial.*
2. $\mathrm{spec}_T(\mathcal{A})$ *is upward closed in the Turing degrees if and only if* $\mathcal{A}$ *is not trivial.*

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A structure $\mathcal{A}$ with universe $\mathbb{N}$ is trivial if there exists a finite subset $S \subset \mathbb{N}$ such that any permutation of $\mathbb{N}$ fixing S pointwise is an automorphism of $\mathcal{A}$.

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II. *What classes of reals can be written as* $\bigcup(\text{spec}_{wtt}(\mathcal{A}))$ *for a structure* $\mathcal{A}$*?*

III. *Just how is a wtt degree spectrum different from a Turing degree spectrum?*

Furthermore, what happens when we narrow the class of structures $\mathcal{A}$ that are allowed?

A result on wtt degree spectra

When we classify the possible Turing degree spectra, the following dichotomy is a good start.

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1. $\text{spec}_{wtt}(\mathcal{A})$ *is a singleton if and only if* $\mathcal{A}$ *is trivial.*
2. $\text{spec}_{wtt}(\mathcal{A})$ *avoids an upward cone if and only if* $\mathcal{A}$ *is w-trivial.*
3. $\text{spec}_{wtt}(\mathcal{A})$ *contains an upward cone if and only if* $\mathcal{A}$ *is not w-trivial.*

Is the wtt case really distinct?

As subsets of $2^{\mathbb{N}}$, it is easy to see that the inequality

$$\bigcup \mathrm{spec}_{wtt}(\mathcal{A}) \subseteq \bigcup \mathrm{spec}_{T}(\mathcal{A})$$

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holds. There are plenty of examples where the two sets are equal:

Proposition

For any nontrivial $\mathcal{B}$, there is an $\mathcal{A}$ such that $\bigcup \mathrm{spec}_{wtt}(\mathcal{A}) = \bigcup \mathrm{spec}_{T}(\mathcal{B})$. In fact, $\mathcal{A}$ can be a graph.

We'd like to be sure that this is not always the case.

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Proposition

1. *If $\mathcal{A}$ is trivial, and its Turing degree consists of more than one wtt degree, then the inclusion is strict.*
2. *For any wtt degree $\mathbf{b}$, we can construct a $\mathcal{B}$, with infinite signature, such that $\text{spec}_{wtt}(\mathcal{B}) = \mathcal{D}_{wtt}(\geq \mathbf{b})$.*

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3. *There exists a nontrivial structure $\mathcal{C}$ with finite signature where the inclusion is strict.*

Structures with finite signature

Theorem (H–K–S–S 2002)

If $\mathcal{B}$ is a nontrivial structure, then there exists a graph $\mathcal{G}$ such that $\mathrm{spec}_T(\mathcal{G}) = \mathrm{spec}_T(\mathcal{B})$.

We say that graphs are *universal for Turing degree spectra.*

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If $\mathcal{A}$ is a structure with finite signature and $\mathcal{A}$ is w-trivial, then $\mathcal{A}$ is trivial. In particular, graphs are not similarly universal for wtt degree spectra.

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Question

Is there an interesting class of structures (for example, graphs) that is universal for wtt degree spectra for models of finite signature?

When is $\text{spec}_{wtt}(\mathcal{A})$ upward closed?

Recall:

Theorem (Knight 86)

$\text{spec}_T(\mathcal{A})$ *is upward closed if and only if* $\mathcal{A}$ *is not trivial.*

It is fairly easy to show that the wtt degree spectrum is upward closed for 'nice' types of structure.

1. Nontrivial equivalence relations
2. Nontrivial graphs with infinitely many components
3. Groups, and so on

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Question

If $\mathrm{spec}_{wtt}(\mathcal{A})$ contains a cone (i.e., if it is not w-trivial), must it be upward closed?

A quick construction

Proposition

For any nontrivial $\mathcal{B}$, there is an $\mathcal{A}$ such that $\bigcup \text{spec}_{wtt}(\mathcal{A}) = \bigcup \text{spec}_T(\mathcal{B})$.

We may assume that $\mathcal{B}$ is a graph.

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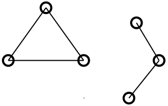

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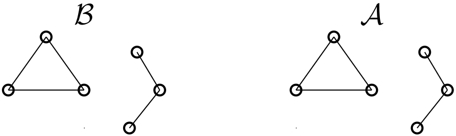

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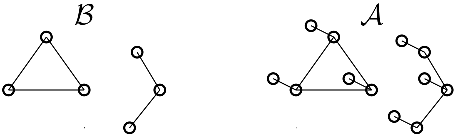

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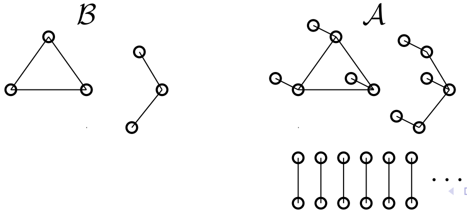

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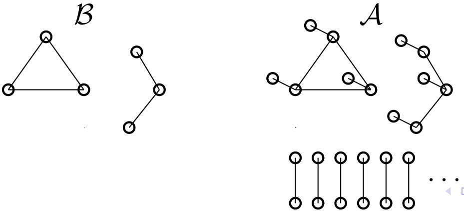

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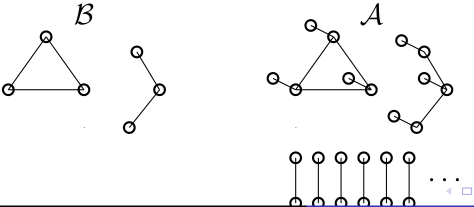

$\square$

Questions

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If $\text{spec}_{wtt}(\mathcal{A})$ *contains a cone, must it be upward closed?*
. . . Or is there some other nice dichotomy to be found?

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Is there an interesting class Δ *of structures such that, for each* $\mathcal{A}$ *with finite signature, there is a* $\mathcal{B} \in \Delta$ *with the same wtt degree spectrum?*
. . . for each $\mathcal{A}$ *with a single binary relation symbol . . . ?*

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Can we characterize the structures $\mathcal{A}$ *such that* $\bigcup \text{spec}_T(\mathcal{A}) = \bigcup \text{spec}_{wtt}(\mathcal{A})$*?*

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Definition

A structure $\mathcal{A}$ with universe A and relations $(R_0, R_1, \ldots)$ is w-trivial if, for each total computable function f, there is a finite set S witnessing the triviality of the reduct of $\mathcal{A}$ to the language $(R_0, R_1, \ldots, R_{f(|S|)})$.