Proving that Artinian implies Noetherian without proving that Artinian implies finite length

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Background

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- 1956: Frölich and Shepherdson give basic definitions; construct an explicitly given field with no splitting algorithm.
- Computable Ring Theory.

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Definition

A *computable ring* is a computable subset $A \subseteq \mathbb{N}$ equipped with two computable binary operations $+$ and $\cdot$ on A, together with elements $0, 1 \in A$ such that $R = (A, 0, 1, +, \cdot)$ is a ring.

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All rings will be *countable* and *commutative*, unless we say otherwise.

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A ring R is *strongly Noetherian* if there exists a number $n \in \mathbb{N}$ such that the length of every strictly increasing chain of ideals in R is bounded by n.

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Reverse Mathematics

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$$\mathsf{ACA}_0 \longrightarrow \mathsf{WKL}_0 \longrightarrow \mathsf{RCA}_0$$

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Mileti: What is the reverse mathematical strength of the theorem that says every Artinian ring is Noetherian?

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If, for every $n \in \mathbb{N}$, R contains a strictly increasing chain of ideals of length n, then R also contains an infinite strictly decreasing chain of ideals.

How much computational power does it take to go from an infinite strictly increasing chain of ideals to an infinite strictly decreasing chain of ideals?

Theorem (Conidis, 2010)

There is a computable integral domain R with an infinite uniformly computable strictly increasing chain of ideals, and such that every strictly decreasing chain of ideals in R contains a member of PA degree.

New Results

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Corollary (RCA_0)

Theorem 1 implies WKL_0.

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4. *If R is Artinian, then the Jacobson radical $J \subset R$ and nilradical $N \subset R$ exist and are equal.*
5. *If R is Artinian, then $J \subset R$ exists and R/J is Noetherian.*

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This ends Part I.

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$$Ann(x) = \{y \in R : x \cdot y = 0\} \subset R$$

The Classical Proof of Theorem 1

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Key Lemma

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- Consider the descending chain of ideals

$$M_1 \supset M_1 M_2 \supset M_1 M_2 M_3 \supset \cdots \supset J = \prod_{k=1}^{n_0} M_k \supset$$

$$\supset M_1 J \supset M_1 M_2 J \supset \cdots \supset J^2 \supset \cdots$$

$$\vdots$$

$$\cdots \supset M_1 J^{N-1} \supset M_1 M_2 J^{N-1} \supset \cdots \supset J^N = 0.$$

- Let $M_0 = R$. For all $1 \leq k \leq N \cdot n_0$, write $k = q_k n_0 + r_k$, $r_k < n_0$, and let

$$W_k = M_1 M_2 \cdots M_{r_k} J^{q_k}.$$

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- Each W_k / W_{k+1} is finite dimensional, since R is Artinian.
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- It was previously known [Conidis, 2010] that, modulo the Key Lemma, this all works in $\mathrm{WKL}_0 + \mathrm{I}\Sigma_2$.

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Theorem 1 is equivalent to WKL_0 over $RCA_0 + I\Sigma_2$.

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Theorem (Conidis, 2012)

The Key Lemma is equivalent to WKL_0 over RCA_0.

A New Proof of the Key Lemma

Lemma (WKL_0)

Let $z_0, z_1, \ldots, z_l \in J \subset R$. Then there exists $N \in \mathbb{N}$ such that every product of degree N with factors $z_0, z_1, \ldots, z_l$ is zero.

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We reason in WKL_0. Let $J = \{z_0, z_1, z_2, \ldots\}$ be an enumeration of J, and for all $l \in \mathbb{N}$ set

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The finite set $\{z_i\}_{0 \leq i \leq l_0}$ essentially witnesses the fact that J is nilpotent.

Structure Theorem for Artinian Rings

Theorem (WKL_0)

Every Artinian ring is the direct product of finitely many local Artinian rings.

Choose $N \in \mathbb{N}$ such that

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Let $\langle N_1, N_2, \ldots, N_{n_0} \rangle \leq \langle N, N, \ldots, N \rangle$ be least such that

$$M_1^{N_1} M_2^{N_2} \cdots M_{n_0}^{N_{n_0}} = 0.$$

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$$M_1^{N_1} M_2^{N_2} \cdots M_{n_0}^{N_{n_0}} = 0.$$

Show that $M_1^{N_1}, M_2^{N_2}, \ldots, M_{n_0}^{N_{n_0}}$ exist.

Structure Theorem for Artinian Rings

Theorem (WKL_0)

Every Artinian ring is the direct product of finitely many local Artinian rings.

Choose $N \in \mathbb{N}$ such that

$$J^N = M_1^N M_2^N \cdots M_{n_0}^N = 0.$$

Let $\langle N_1, N_2, \ldots, N_{n_0} \rangle \leq \langle N, N, \ldots, N \rangle$ be least such that

$$M_1^{N_1} M_2^{N_2} \cdots M_{n_0}^{N_{n_0}} = 0.$$

Show that $M_1^{N_1}, M_2^{N_2}, \ldots, M_{n_0}^{N_{n_0}}$ exist.
Use Chinese Remainder Theorem to get that

$$R \cong R/M_1^{N_1} \times R/M_2^{N_2} \times \cdots \times R/M_{n_0}^{N_{n_0}}.$$

References

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