Combinatorial invariants and weak equivalence

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Part I

Introduction

Many classical dichotomy theorems in descriptive set theory can be cast in the setting of definable graphs and their combinatorial parameters, and these graphs are unavoidable in the study of Borel equivalence relations.

From a more ergodic-theoretic point of view, these combinatorial invariants can also yield information about the global dynamics of group actions.

In this talk we focus on the second aspect, and more specifically on the relationship between graph-theoretic invariants and weak equivalence of probability measure preserving actions of a countable group.

This is joint work with Alexander S. Kechris and Robin D. Tucker-Drob.

Part II

Graph theory

Definition

A *graph* G on a set X is a symmetric, irreflexive subset of X^2.

Definition

A set $A \subseteq X$ is $(G\text{-})$*independent* if $G \cap A^2 = \emptyset$.

Definition

A function $c : X \to Y$ is a $(Y\text{-})$*coloring* of G if $c^{-1}(\{y\})$ is G-independent for each $y \in Y$.

For a graph G on finite vertex set X we have the following familiar numbers.

Definition

The *independence ratio* of G, denoted by $i(G)$, is given by $\max\left\{\frac{|A|}{|X|} : A \subseteq X \text{ is } G\text{-independent}\right\}$.

Definition

The *chromatic number* of G, denoted by $\chi(G)$, is given by $\min\{n : \text{there is an } n\text{-coloring of } G\}$.

Remark

Since each color is independent, we have $i(G)\chi(G) \geq 1$.

These definitions suggest the following analogs for a Borel graph G on a standard probability space (X, μ).

Definition

The *independence number* of G, denoted by $i_\mu(G)$, is given by $\sup\{\mu(A) : A \subseteq X \text{ is Borel and } G\text{-independent}\}$.

Definition

The (μ-)*measurable chromatic number* of G, denoted by $\chi_\mu(G)$, is given by $\min\{|Y| : Y \text{ is standard Borel and there is a } \mu\text{-measurable } Y\text{-coloring of } G\}$.

Remark

So $i_\mu(G) \in [0, 1]$ and $\chi_\mu(G) \in \{1, \ldots, \aleph_0, 2^{\aleph_0}\}$.

Remark

Since each color is independent, we have $i_\mu(G)\chi_\mu(G) \geq 1$.

Remark

There is a variation on the chromatic number that interacts better with weak containment of group actions.

Definition

The (μ-)*approximate chromatic number* of G, written $\chi_\mu^{\mathrm{ap}}(G)$, is the least cardinality of a standard Borel space Y such that for all $\varepsilon > 0$ there is a Borel set $A \subseteq X$ with $\mu(A) > 1 - \varepsilon$ and $G \cap A^2$ Y-colorable by a μ-measurable function.

Remark

Certainly $\chi_\mu^{\mathrm{ap}}(G) \leq \chi_\mu(G)$.

Remark

We still have $i_\mu(G)\,\chi_\mu^{\mathrm{ap}}(G) \geq 1$.

Example

It is sometimes the case that $\chi_\mu^{\mathrm{ap}}(G) < \chi_\mu(G)$. For example, if $\sigma : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ is the shift map, $\sigma(x)(n) = x(n-1)$, $X \subseteq 2^{\mathbb{Z}}$ is the set of points which have infinite σ-orbits, μ is the product measure, and G is the graph relating points $x, y \in X$ iff $x = \sigma(y)$ or $y = \sigma(x)$, then $\chi_\mu^{\mathrm{ap}}(G) = 2$ but $\chi_\mu(G) = 3$. Note that (in ZFC) this graph has ordinary chromatic number 2 since it is acyclic.

Part III

Group actions

The previous example is a prototype of the more general situation which we will investigate: graphs associated with free measure-preserving actions of finitely generated groups.

Definition

Suppose that Γ is a group with finite generating set S (assumed hereafter to be symmetric), and a is an action of Γ by μ-preserving Borel automorphisms on a standard probability space (X, μ). We define the graph $G(\Gamma, a, S)$ on X by relating two points $x, y \in X$ if $x \neq y$ and there exists $s \in S$ with $y = s^a(x)$. We sometimes abbreviate $G(\Gamma, a, S)$ by $G(a)$.

Remark

If the action a is free, then each connected component of $G(\Gamma, a, S)$ is isomorphic to the Cayley graph of Γ with respect to S.

Theorem

Suppose that Γ is an infinite group with finite generating set S and a is a free, μ-preserving action of Γ on (X, μ). Then $\chi_\mu^{\mathrm{ap}}(G(a)) \leq |S|$ and thus $i_\mu(G(a)) \geq 1/|S|$.

Remark

In the special case that Γ has finitely many ends and is isomorphic neither to $\mathbb{Z}$ nor to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, the above conclusion may be improved to $\chi_\mu(G(a)) \leq |S|$ (and in fact further to a Borel $|S|$-coloring of G).

Remark

The theorem in fact holds in greater generality: if G is any Borel graph on (X, μ) such that each point has finite degree at most d, then $i_\mu(G) \geq 1/d$ and $\chi_\mu^{\mathrm{ap}}(G) \leq d$.

We next discuss the relationship between these combinatorial notions and weak equivalence of group actions. For convenience we denote by $\mathrm{FR}(\Gamma, X, \mu)$ the space of free, μ-preserving actions of a group Γ on a standard probability space (X, μ).

Definition

For $a, b \in \mathrm{FR}(\Gamma, X, \mu)$, we say that a is *weakly contained* in b, written $a \prec b$, if for any measurable sets $A_1, \ldots, A_n \subseteq X$, any finite set $F \subseteq \Gamma$, and any $\varepsilon > 0$, there are measurable sets $B_1, \ldots, B_n \subseteq X$ such that

$$|\mu(\gamma^a(A_i) \cap A_j) - \mu(\gamma^b(B_i) \cap B_j)| < \varepsilon,$$

for any $\gamma \in F$ and $i, j \leq n$.

Remark

Equivalently, $a \prec b$ exactly when a is in the weak closure of the conjugacy class of b.

Definition

We say that actions a and b are *weakly equivalent*, written $a \sim b$, if $a \prec b$ and $b \prec a$.

Theorem

Suppose that Γ is a group with finite generating set S. Suppose that $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ with $a \prec b$. Then $i_\mu(G(a)) \leq i_\mu(G(b))$, and $\chi_\mu^{\mathrm{ap}}(G(a)) \geq \chi_\mu^{\mathrm{ap}}(G(b))$.

Theorem

Suppose Γ is an infinite group with finite generating set S such that the Cayley graph of Γ with respect to S is bipartite. Then the set $\{i_\mu(G(a)) : a \in \mathrm{FR}(\Gamma, X, \mu)\}$ is a closed interval $[\alpha, 1/2]$ for some $\alpha \geq 1/|S|$. Moreover, $\alpha = 1/2$ exactly when Γ is amenable.

Question

What is the spectrum of possible independence numbers of *ergodic* actions of Γ?

Remark

This characterization of amenability by having a unique independence number may fail if the Cayley graph of Γ is not bipartite. For example, every free, measure-preserving action of $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ with the standard generating set has independence number $1/3$.

While χ_μ^{ap} is invariant across a weak equivalence class of Γ-actions, χ_μ need not be. Surprisingly, we can "un-approximate" the approximate chromatic number without leaving a weak equivalence class.

Theorem

Suppose that Γ is a finitely generated group and $a \in \mathrm{FR}(\Gamma, X, \mu)$. Then there is some $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \sim a$ and $\chi_\mu(G(b)) = \chi_\mu^{\mathrm{ap}}(G(a))$.

Theorem

Similarly, there is some $b \sim a$ and $A \subseteq X$ Borel such that A is $G(b)$-independent and $\mu(A) = i_\mu(G(a))$.

Part IV

Applications to probability theory

Definition

A *random k-coloring* of a graph G on a countable set X is a Borel probability measure on the space of k-colorings of G, viewed as a closed subset of k^X.

Definition

A *translation-invariant* random k-coloring of the Cayley graph of Γ with respect to S is one which is invariant under the action of Γ on the space of k-colorings induced by translations of the Cayley graph.

Remark

There is a natural correspondence between μ-measurable colorings of free μ-preserving actions of Γ on (X, μ) and translation-invariant random k-colorings of the Cayley graph of Γ.

Remark

If Γ is amenable with k-colorable Cayley graph, then there is translation-invariant random k-coloring of the Cayley graph, since the space of k-colorings forms a nonempty compact set on which Γ acts by homeomorphisms.

Remark

More degenerately, if Γ has bipartite Cayley graph, then there is a translation-invariant random 2-coloring of the Cayley graph, since there's an invariant measure for any action on a two point set.

Theorem (Schramm, indep. Kechris-Solecki-Todorcevic)

There is a translation-invariant random $(|S|+1)$-coloring of the Cayley graph of Γ.

Question (Aldous-Lyons)

If Γ is infinite, does its Cayley graph admit a translation-invariant random $|S|$-coloring?

Answer

Yes! In fact, if Γ has finitely many ends, we can even find a random $|S|$-coloring invariant under the full automorphism group of the Cayley graph.

Question

Are there Γ, k for which the Cayley graph of Γ admits a k-coloring but not a translation-invariant random k-coloring?

Part V

Thanks!