A Random Turing degree

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based on joint work with George Barmpalias and Andrew Lewis

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What is a random Turing degree?

- Consider the structure of the Turing degrees, $\mathcal{D}$. Given $\mathbf{a} \in \mathcal{D}$ we can ask what properties hold of $\mathbf{a}$.
- Properties expressible in 1st order logic with $\leq$ e.g. $\mathbf{a}$ is a minimal degree.
- Other properties e.g. $\mathbf{a}$ bounds a 1-generic degree.

Measurability of sets of degrees

Fix some property P.

- Consider the set $S = \{X \subseteq \omega \mid P \text{ holds of } \deg(X)\}$.
- S is a tailset (i.e. $\sigma X \in S \Rightarrow X \in S$) hence by Kolmogorov's 0-1 law $\mu(S) = 0$ or $\mu(S) = 1$ **provided that S is measureable.**

Definition (attempt)

Call $\mathbf{a} \in \mathcal{D}$ a *random Turing degree* if $\mathbf{a}$ is a member of all definable (without parameters) sets of Turing degrees of measure 1.

Independence

Question

Which properties of of Turing degrees are measurable? In particular, are all definable sets of Turing degrees measurable?

Lemma

The statement "All definable sets of Turing degrees are measurable" is independent of ZFC.

- ZFC + PD $\Rightarrow$ All definable sets of Turing degrees are measurable.
- ZFC + V=L $\Rightarrow$ There exists a non-measurable definable set of Turing degrees.

Previous results

Note properties, expressible in first order logic with $\leq$, restricted to a lower cone e.g. $\mathcal{D}(\mathbf{a}, \leq_T)$ are always measurable.

Property	Measure	Due to
Is minimal	0	Sacks (1963)
Bounds a minimal degree	0	Paris (1977)
1-generics are downwards dense	1	Kurtz (1981)
Is c.e.a.	1	Kurtz (1981)
Has a strong minimal cover	1	Barmpalias-Lewis (2011)

Algorithmic randomness

Definition

- A set A is called X-random, if for every X-computable sequence of open sets $\{U_i\}_{i \in \omega}$, such that $\mu U_i \leq 2^{-i}$,

$$A \notin \bigcap_i U_i.$$

- A set A is 1-random if it is $\emptyset$-random.
- A set A is 2-random if it is $\emptyset'$-random.

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Kautz (1991) investigated if what level of algorithmic randomness is sufficient to ensure the above conditions. He showed every 2-random bounds a 1-generic and every 2-random is c.e.a.

New results

Theorem (Barmpalias-D-Lewis)

The 1-generic degrees are downwards dense below any 2-random degree.

Corollary

No 2-random degree bounds a minimal degree.

This result is optimal because there are Demuth random degrees and weakly 2-random degrees that bound minimal degrees.

Aspects of proof

Given a Turing functional Θ.

Build a Turing functional Φ such that:

$$\begin{array}{ll} X & \text{2-random} \\ \big\downarrow \Theta & \\ Y & \text{noncomputable} \\ \big\downarrow \Phi & \\ Z & \text{1-generic} \end{array}$$

Really build a family of functionals $\Phi_1, \Phi_2, \ldots$ such that

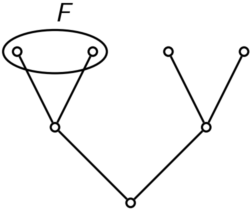
$$\mu\{X \colon \Phi_i(\Theta(X)) \text{ is total}\} \geq 1 - 2^{-i}.$$

Hence if X is 2-random, some Φ_i is total with oracle Θ^X.

Aspects of proof – Constructing Φ_1

Would like: for all Y, if Φ_1^Y is total then Φ_1^Y is 1-generic.

Ensuring Φ_1^Y meets or avoids W_1 the first c.e. set.



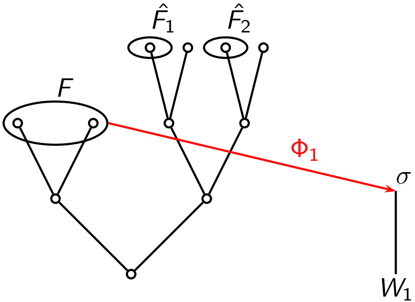
Find F so the that $\delta \leq \mu\{X\colon \Theta^X \succeq [F]\} \leq 1/4$.

Restrain definition of Φ_1 on elements of $[F]$ unless some σ enters W_1. If no string enters W_1, then all elements in the complement of $[F]$ have meet this requirement.

Aspects of proof – Constructing Φ_1.

If some σ enters W_1, then:

- Define $\Phi_1^X \succeq \sigma$ for all $X \in [F]$.
- Attempt to meet this requirement for some paths in the complement of $[F]$.



The functional Φ_1 is restrained on extensions of $[\hat{F}]$ until some compatible extension enters W_1.

Undecidability

Theorem (Greenberg-Montalbán)

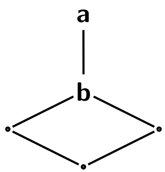
If $\mathbf{a}$ is a degree such that the degrees containing 1-generic sets are downwards dense below $\mathbf{a}$, then the theory of $\mathcal{D}(\mathbf{a}, \leq)$ interprets true first-order arithmetic.

Corollary

If $\mathbf{a}$ contains a 2-random set, then the theory of $\mathcal{D}(\mathbf{a}, \leq)$ interprets true first-order arithmetic.

Strong minimal covers

A degree $\mathbf{a}$ is a strong minimal cover if there exists a degree $\mathbf{b} < \mathbf{a}$, such that for all $\mathbf{c} < \mathbf{a}$, $\mathbf{c} \leq \mathbf{b}$:

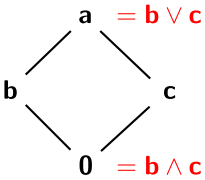


Theorem (Barmpalias-D-Lewis)

- *No degree below a 2-random is a strong minimal cover.*
- *Every degree below a 2-random has a strong minimal cover.*

Top of a diamond

A degree $\mathbf{a}$ is the top of a diamond if there exist degrees $\mathbf{b}, \mathbf{c}$ such that the following diagram holds:



Theorem (Barmpalias-D-Lewis)

Every non-zero degree below a 2-random is the top of a diamond.

Distribution of random sets

Question

How are the n-random sets distributed in the Turing degrees?

Question

For $n \geq 2$, is there an $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{D}$ such that:

1. $\mathbf{a} < \mathbf{b} < \mathbf{c}$.
2. $\mathbf{a}$ and $\mathbf{c}$ both contain n-random sets.
3. $\mathbf{b}$ does not contain an n-random set.

Conditions for a Turing degree not to be 2-random are usually either upwards or downwards closed.

Question

What is the measure of the set $\{X \in 2^\omega \mid \deg(X) \text{ is a minimal cover}\}$?