The set-theory of Compact spaces and converging sequences and stuff

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April 1, 2012

[1] and Statistics

introduction

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Converging sequences in this talk will come in two flavors

$\omega+1$ means converging sequence with limit

ω_1+1 means co-countably converging

G_δ-points, Frechet

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building up more complicated spaces

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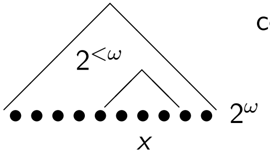

compact metric

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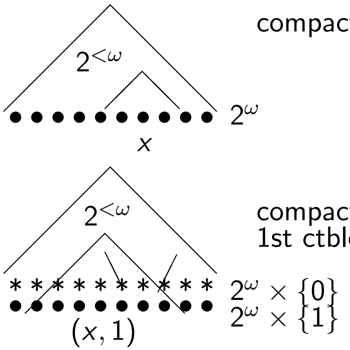

compact metric

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1st ctble

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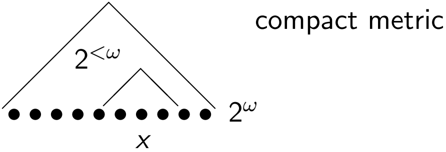

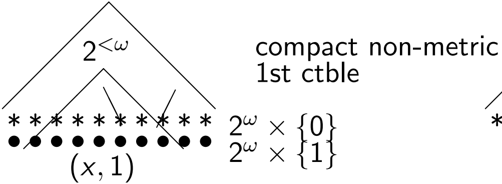

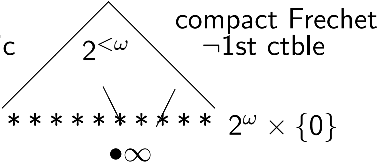

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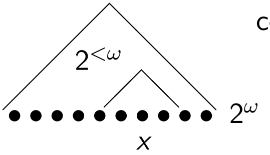

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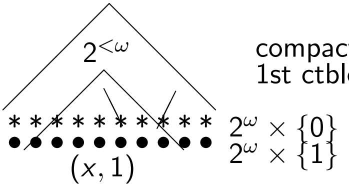

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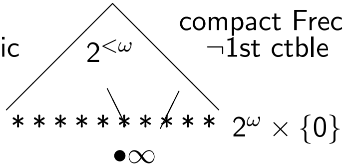

compact Frechet
$\neg$1st ctble

$\Leftarrow$ blow up point

to Cantor set to get $\chi = \omega$

converging sequences but not Frechet

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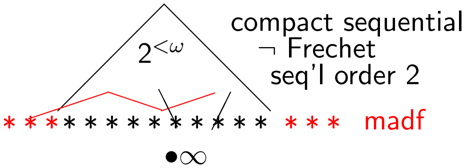

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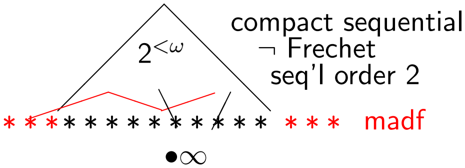

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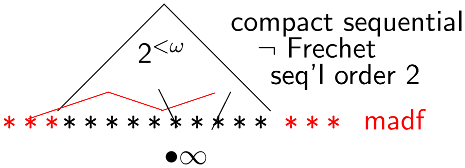

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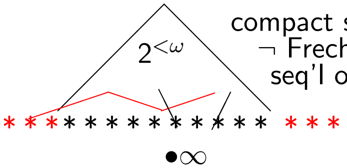

compact sequential
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seq'l order 2

madf

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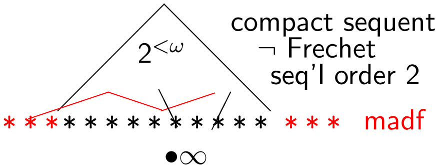

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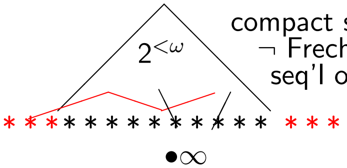

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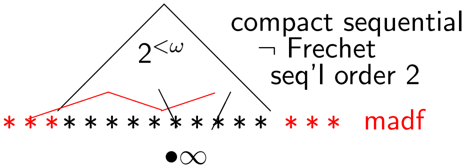

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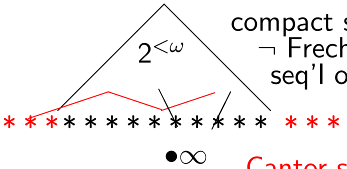

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Cantor set blow-up fails

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so, oddly, containing a converging ω_1-sequence is a largeness property (recall $\beta\mathbb{N} \supset \omega_1{+}1$)

dichotomies and questions

Efimov

does each compact space contain one of $\omega{+}1$ or $\beta\omega$?

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celebrated Moore-Mrowka

Is every compact space of countable tightness also sequential?

Say that an Efimov space is a compact space containing neither $\omega + 1$ nor $\beta\omega$.

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Say that a Moore-Mrowka space is a compact $t = \omega$ space which is not sequential.

Čech-Pospišil labelled trees – an object needing analysis

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For a tree $T \subset 2^{<\mathfrak{c}}$, we may attach a clopen set a_t, we will (arrange and) let $F_t = \bigcap_{s \leq t} a_s \neq \emptyset$, and we might pick a point $x_t \in F_t$. We extend t if $|F_t| > 1$; in addition $\{a_{t0}, a_{t1}\}$ will be disjoint (and sometimes a partition).

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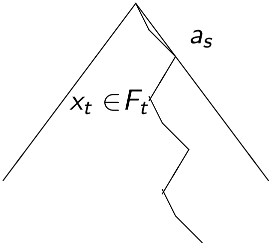

F_t is a $G_{|t|}$-set
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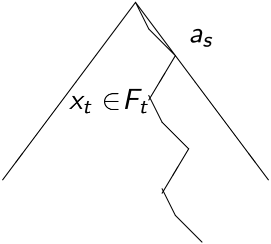

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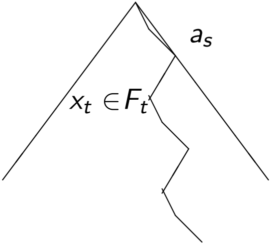

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CH implies every compact $t = \omega$ space has a $G_{\leq\omega_1}$-point.

PFA and Moore-Mrowka

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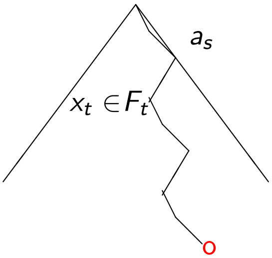

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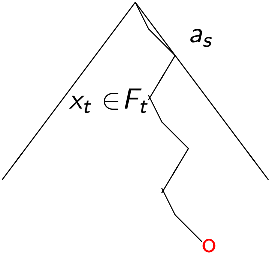

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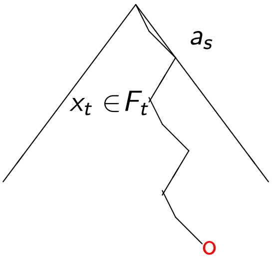

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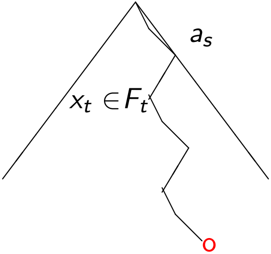

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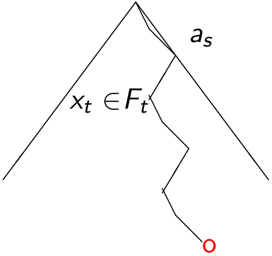

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this shows we can't control this tree
(general "Čech-Pospišil" tree)

but we have to.

remark: It may not be fully branching

Fedorchuk / Ostaszewski embrace the tree

Definition (Koszmider / Koppelberg / Fedorchuk)

A T-algebra will be a Boolean algebra with a generating set indexed by a binary tree T, i.e. $\{a_t : t \in T\}$ and a_{t0}, a_{t1} are complements. And more properties in a minute.

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$\diamondsuit$ implies there is a Moore-Mrowka space not containing $\omega{+}1$.

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Proof.

Build a suitable T-algebra and take the Stone space. □

T-algebras ; their Stone spaces do not contain $\beta\mathbb{N}$

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for each $b \in bT$, $\{a_t : t \leq b\}$ generates a superatomic Boolean algebra; and every superatomic Boolean algebra can be expressed as a maximal branch in a T-algebra

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for each $b \in bT$, $\{a_t : t \leq b\}$ generates a superatomic Boolean algebra; and every superatomic Boolean algebra can be expressed as a maximal branch in a T-algebra

the key is that a_t can not "split F_s" if $s \perp t$

T-algebras ; their Stone spaces do not contain $\beta\mathbb{N}$

For a tree T, let bT denote T together with all its maximal branches. A family $\{a_t : t \in T\}$ is a T-generating family if

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- New result: $\mathfrak{b} = \mathfrak{c}$ implies an Efimov T-algebra exists (joint with Shelah) ; Not previously known even for just Efimov

a detour then back to Moore-Mrowka

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A space X is initially ω_1-compact if every set of size $\leq\omega_1$ has a complete accumulation point.

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[ZFC] Any compactification of a non-compact initially ω_1-compact $t = \omega$ space is a Moore-Mrowka space.

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Proposition

Mixed cardinality elementary submodels as side conditions give a proper poset of finite conditions that add the Baumgartner-Shelah example.

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These T-algebra chain examples are intrinsically of size $\mathfrak{c}$ and MA certainly does not hold. Can the Neeman method be used?

Neeman's method and a new example

Let $T \subset 2^{<\omega_2}$, and use Velickovic approach of mixed cardinality finite ϵ-chains of elementary submodels to define a poset $\mathbb{P}_T$ adding a Rabus style T-algebra

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A condition p consists of $(H_p, \{a^p_t : t \in H_p\}, \mathcal{M}_p)$

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4. $\{a_t : t \in M \cap H_p\}$ generates a subalgebra $\subset M$ for each $M \in \mathcal{M}_p$

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The set X of all ultrafilters from branches with countable cofinality is first countable and dense in $S(B_T)$. If T has no cofinality ω_1-branches, X is initially ω_1-compact.

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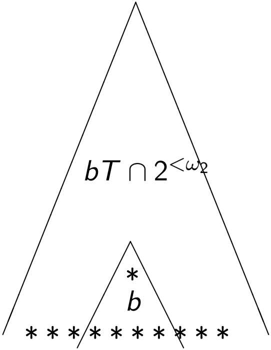

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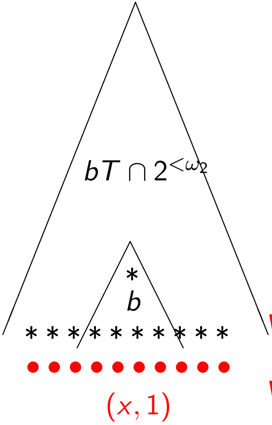

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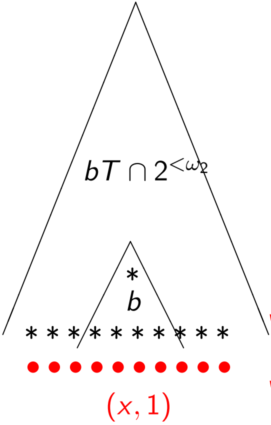

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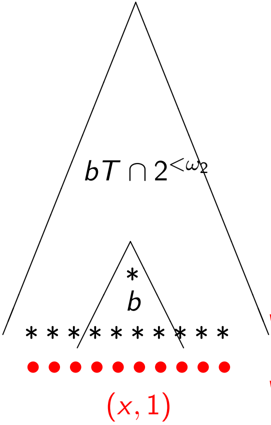

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and (even before the forcing) has $\aleph_1$-sized sets not contained in an $\aleph_1$-sized Lindelof subset

Husek question about small diagonal

A compact space X has a small diagonal if the quotient space X^2/Δ_X contains no converging ω_1-sequence.

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In many models (PFA, CH, Cohen) each compact space with a small diagonal is metrizable. Is this true in ZFC?

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Questions

We do not know if X contains $\omega+1$, points of countable character, has cardinality at most $\mathfrak{c}$, and all other metric type properties.

models for compact small diagonal is metrizable

A powerful consequence of not containing any converging ω_1-sequences emerges

Theorem

In any model obtained by FS iteration of small σ-linked posets a compact space X contains no converging ω_1-sequences iff it is first-countable and Lindelof sets are stationary in $[X]^{\aleph_1}$.

It follows that compact spaces of small diagonal are metrizable in such a model.

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Lemma

A key step was from a Junqueira-Koszmider paper showing that forcing with such posets preserve that compact spaces stay Lindelof in the extension.

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7. Scarborough-Stone: is the product of (all) sequentially compact spaces still countably compact