Computable Mathias genericity

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On Mathias generic sets.

Joint work with Peter A. Cholak and Jeffry L. Hirst.

How the world computes, Lecture Notes in Computer Science, to appear.

Mathias conditions

Definition.

1 A (computable Mathias) pre-condition is a pair (D, E) such that D is a finite set, E is a computable set, and $\max D < \min E$.

2 (D, E) is a (computable Mathias) condition if E is infinite.

3 A pre-condition (D', E') extends (D, E), written $(D', E') \leqslant (D, E)$, if $D \subseteq D' \subseteq D \cup E$ and $E' \subseteq E$.

4 A set S satisfies (D, E) if $D \subseteq S \subseteq D \cup E$.

Named after Mathias's use of it in set theory, but used earlier by Soare and others in computability theory.

Useful in studying Ramsey's theorem and related properties. In computability, used in various arguments about RT^2_2.

Mathias generics

A set S meets a set $\mathcal{C}$ of conditions if it satisfies some condition in $\mathcal{C}$.

S avoids $\mathcal{C}$ of conditions if it meets the conditions with no extension in $\mathcal{C}$.

Definition.

1. A Σ^0_n set of conditions is a Σ^0_n-definable set of pre-conditions, each of which is a condition.
2. A set G is (Mathias) n-generic if it meets or avoids every Σ^0_n set of conditions.
3. A set G is weakly (Mathias) n-generic if it meets every dense Σ^0_n set of conditions.

Computable setting

Definition. An index for a pre-condition (D, E) is a pair $(d, e) \in \omega^2$ such that d is the canonical index of D and $E = \{x \in \omega : \varphi_e(x) \downarrow = 1\}$.

The set of all (indices for) pre-conditions is Π^0_1, but this has a computable subset containing an index for every pre-condition.

Even working over this set, the set of all (indices for) conditions is Π^0_2.

Definition. A set G is strongly (Mathias) n-generic if it meets or avoids every Σ^0_n-definable set of pre-conditions.

Proposition (Cholak, Dzhafarov, Hirst). A set is strongly n-generic if and only if it is $\max\{n, 3\}$-generic.

Without further comment, n below will always be a number $\geqslant 3$.

Comparison with Cohen generics

Computability of Cohen generics studied by Jockusch, Kurtz, and others.

Similarities.

1 Implications: n-generic $\implies$ weakly n-generic $\implies$ $(n-1)$-generic.

2 There exists an n-generic $G \leqslant_T \emptyset^{(n)}$.

3 Every weakly n-generic set is hyperimmune relative to $\emptyset^{(n-1)}$.

Dissimilarities.

1 Every weakly Mathias n-generic set G is cohesive. Hence, if $G = G_0 \oplus G_1$ then either $G_0 =^* \emptyset$ or $G_1 =^* \emptyset$.

2 If G is Mathias 3-generic then $G' \geqslant \emptyset''$.

Thus, no Mathias n-generic can be Cohen 1-generic, and no Cohen 2-generic can even compute a Mathias 3-generic.

Jump properties

It is a well-known result of Jockusch that if G is Cohen n-generic then $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$. In particular, every Cohen generic set has $\mathbf{GL}_1$ degree.

Theorem (Cholak, Dzhafarov, Hirst). If G is Mathias n-generic, then:

1. $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$;
2. G has $\mathbf{GH}_1$ degree, i.e., $G' \equiv_T (G \oplus \emptyset')'$.

Corollary. If G is Mathias n-generic then it has $\overline{\mathbf{GL}_1}$ degree. So G cannot have Cohen 1-generic degree, but G computes a Cohen 1-generic.

Complexity of the forcing relation

Let L_1^* be the language of first-order arithmetic, with a special set variable, X, and the epsilon relation, $\in$. Let $\varphi(X)$ be a formula of L_1^*.

We can define the forcing relation $(D, E) \Vdash \varphi(G)$ inductively such that forcing implies truth:

Proposition (Cholak, Dzhafarov, Hirst). If φ is Σ_n^0, and if G is n-generic and satisfies some (D, E) that forces $\varphi(G)$, then $\varphi(G)$ holds.

Lemma (Cholak, Dzhafarov, Hirst).

1. If φ is Σ_0^0, then the relation $(D, E) \Vdash \varphi(G)$ is computable.
2. If φ is Π_1^0, Σ_1^0, or Σ_2^0, then so is the relation $(D, E) \Vdash \varphi(G)$.
3. If φ is Π_n^0 for some $n \geqslant 2$, then the relation $(D, E) \Vdash \varphi(G)$ is Π_{n+1}^0.
4. If φ is Σ_n^0 for some $n \geqslant 3$, then the relation $(D, E) \Vdash \varphi(G)$ is Σ_{n+1}^0.

Computing from Mathias generics

So far: Cohen n-generics do not compute Mathias n-generics, but Mathias n-generics compute Cohen 1-generics.

This raises the following question:

Question. Does every Mathias n-generic computes a Cohen n-generic?

Theorem (Cholak, Dzhafarov, Hirst). If G is Mathias n-generic and $B \leqslant_T \emptyset^{(n-1)}$ is bi-immune, then $G \oplus B$ computes a Cohen n-generic.

Thus, for example, by a result of Jockusch, if G is Mathias n-generic then $G \oplus B$ computes a Cohen n-generic for any $\emptyset <_T B \leqslant_T \emptyset'$.

Bi-immune coding

The difficulty with coding into Mathias generics is that if (D, E) is a condition then E can be made very sparse. In particular, it might wipe out a computable set of coding locations.

But if B is bi-immune, then B and $\overline{B}$ must intersect E infinitely often.

Definition. For a finite set $S = \{a_0 < a_1 < \cdots\}$, define

$$S_B = B(a_0)B(a_1)\cdots ,$$

so that $S_B \in 2^{<\omega}$ if S is finite, and $S_B \in 2^{\omega}$ if S is infinite.

Proving the coding theorem

Proof of theorem. Fix a bi-immune $B \leqslant_T \emptyset^{(n-1)}$, and a Σ^0_n set $\mathcal{W} \subseteq 2^{<\omega}$.

Let $\mathcal{C}$ be set of all conditions (D, E) such that D_B belongs to $\mathcal{W}$. Then $\mathcal{C}$ is Σ^0_n, so if G is Mathias n-generic it meets or avoids $\mathcal{C}$.

If G meets $\mathcal{C}$ then G_B meets $\mathcal{W}$.

If G avoids $\mathcal{C}$, then G_B must avoid $\mathcal{W}$. For if G satisfies (D, E) and D_B has an extension τ in $\mathcal{W}$, then we can pass to a finite extension (D', E') of (D, E) such that $D'_B = \tau$.

We conclude that G_B is Cohen n-generic.

No m-reducibility

Proposition (Cholak, Dzhafarov, Hirst). No Mathias n-generic m-computes a Cohen n-generic.

Proof. Let f be a computable function, G a Mathias n-generic, and H a Cohen n-generic, and suppose $f(H) \subseteq G$ and $f(\overline{H}) \subseteq \overline{G}$.

The set of conditions (D, E) with $E \subseteq \operatorname{ran}(f)$ is Σ^0_3, and must be met by G else $G \cap \operatorname{ran}(f)$ would be finite.

So fix such a condition (D, E) that is met by G. Then for all $a > \max D$, $a \in G$ if and only if $a \in E$ and $f^{-1}(a) \subseteq H$.

Thus, $G \leqslant_T H$, meaning $G \equiv_T H$, which cannot be.

Questions

Does every Mathias *n*-generic compute a Cohen *n*-generic?

What is the reverse mathematical content of the principle asserting the existence, for every *X*, of an *n*-generic set for *X*-computable Mathias forcing? It is Π^1_1 conservative over RCA_0, but how about over $B\Sigma^0_2$?

Shore has asked if there are any interesting degrees realizing properties of the form $\mathbf{d}^j = (\mathbf{d}^k \vee \mathbf{0}^l)^m$. The Cohen and Mathias generics realize two such properties. Do generics for other forcing notions realize others?