Generalization by Collapse

Monroe Eskew

University of California, Irvine

meskew@math.uci.edu

March 31, 2012

Introduction

Our goal is to describe a few theorems about small cardinals like ω, ω_1, ω_2, whose proofs are specific to these cardinals.

Then we will show how they can be generalized to larger cardinals using a common method.

Introduction

Our goal is to describe a few theorems about small cardinals like ω, ω_1, ω_2, whose proofs are specific to these cardinals.

Then we will show how they can be generalized to larger cardinals using a common method.

Our theorems concern ideals. I is an ideal on X if it is a collection of subsets of X closed under taking subsets and pairwise unions.

We will only consider ideals which are proper and uniform, meaning $I \neq \mathcal{P}(X)$, and if $A \subseteq X$ and $|A| < |X|$, then $A \in I$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.
- $\mathcal{P}(X)/I$ is the preorder $(I^+, \leq_I)$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.
- $\mathcal{P}(X)/I$ is the preorder $(I^+, \leq_I)$.
- $I \restriction A$ is the smallest ideal containing $I \cup \{X \setminus A\}$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.
- $\mathcal{P}(X)/I$ is the preorder $(I^+, \leq_I)$.
- $I \restriction A$ is the smallest ideal containing $I \cup \{X \setminus A\}$.
- I is κ-complete if it is closed under unions of size $< \kappa$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.
- $\mathcal{P}(X)/I$ is the preorder $(I^+, \leq_I)$.
- $I \restriction A$ is the smallest ideal containing $I \cup \{X \setminus A\}$.
- I is κ-complete if it is closed under unions of size $< \kappa$.
- I is κ-saturated if every antichain in $\mathcal{P}(X)/I$ has size $< \kappa$.

Notions and Notation

Let $I \subseteq \mathcal{P}(X)$ be an ideal.

- $I^* = \{X \setminus A : A \in I\}$, the collection of "$I$-measure-one sets."
- I is prime if I^* is an ultrafilter.
- $I^+ = \mathcal{P}(X) \setminus I$, the collection of "$I$-positive sets."
- For $A, B \subseteq X$, we say $A \leq_I B$ iff $A \setminus B \in I$.
- $\mathcal{P}(X)/I$ is the preorder $(I^+, \leq_I)$.
- $I \restriction A$ is the smallest ideal containing $I \cup \{X \setminus A\}$.
- I is κ-complete if it is closed under unions of size $< \kappa$.
- I is κ-saturated if every antichain in $\mathcal{P}(X)/I$ has size $< \kappa$.
- I is κ-dense if $\mathcal{P}(X)/I$ has a dense subset of size $\leq \kappa$.

Gitik-Shelah Theorem

Theorem (Gitik-Shelah)

There is no countably complete, ω-dense, nowhere-prime ideal.

Gitik-Shelah Theorem

Theorem (Gitik-Shelah)

There is no countably complete, ω-dense, nowhere-prime ideal.

Proof Overview:

- Reduce to the case of a cardinal κ carrying a κ-complete, ω-dense, nowhere-prime ideal.

Gitik-Shelah Theorem

Theorem (Gitik-Shelah)

There is no countably complete, ω-dense, nowhere-prime ideal.

Proof Overview:

- Reduce to the case of a cardinal κ carrying a κ-complete, ω-dense, nowhere-prime ideal.
- Adding a Cohen real g produces an elementary embedding $j : V \to M \subseteq V[g]$, where M is a well-founded ultrapower of V.

Gitik-Shelah Theorem

Theorem (Gitik-Shelah)

There is no countably complete, ω-dense, nowhere-prime ideal.

Proof Overview:

- Reduce to the case of a cardinal κ carrying a κ-complete, ω-dense, nowhere-prime ideal.
- Adding a Cohen real g produces an elementary embedding $j : V \to M \subseteq V[g]$, where M is a well-founded ultrapower of V.
- There is a 1-1 $f : \kappa \to \mathbb{R}$ which is forced to represent g in the ultrapower.

Gitik-Shelah Theorem

Theorem (Gitik-Shelah)

There is no countably complete, ω-dense, nowhere-prime ideal.

Proof Overview:

- Reduce to the case of a cardinal κ carrying a κ-complete, ω-dense, nowhere-prime ideal.
- Adding a Cohen real g produces an elementary embedding $j : V \to M \subseteq V[g]$, where M is a well-founded ultrapower of V.
- There is a 1-1 $f : \kappa \to \mathbb{R}$ which is forced to represent g in the ultrapower.
- For each $\alpha < \kappa^+$, there is a Cohen name τ_α forced to be $j(f)(\alpha)$.

Gitik-Shelah Theorem

Cohen Forcing Fact

Each Cohen term τ canonically codes a Borel function $B(\tau) : \mathbb{R} \to \mathbb{R}$ such that $p \Vdash \tau_1 \neq \tau_2$ iff $\{x : B(\tau_1)(x) = B(\tau_2)(x)\} \cap [p]$ is meager.

Gitik-Shelah Theorem

Cohen Forcing Fact

Each Cohen term τ canonically codes a Borel function $B(\tau) : \mathbb{R} \to \mathbb{R}$ such that $p \Vdash \tau_1 \neq \tau_2$ iff $\{x : B(\tau_1)(x) = B(\tau_2)(x)\} \cap [p]$ is meager.

A contradiction is derived by showing that there are $\alpha < \beta < \kappa^+$ such that $B(\tau_\alpha)$ and $B(\tau_\beta)$ agree on a nonmeager set. $\square$

Woodin's Theorem

Theorem (Woodin)

If there is a countably complete, ω_1-dense ideal on ω_2, then CH holds.

Woodin's Theorem

Theorem (Woodin)

If there is a countably complete, ω_1-dense ideal on ω_2, then CH holds.

Proof Overview:

- Let I be as hypothesized. Forcing with $\mathcal{P}(\omega_2)/I$ gives a generic ultrafilter H on ω_2, and also a generic ultrafilter G on ω_1. We get well-founded ultrapowers $M \cong V^{\omega_2}/H$ and $N \cong V^{\omega_1}/G$.

Woodin's Theorem

Theorem (Woodin)

If there is a countably complete, ω_1-dense ideal on ω_2, then CH holds.

Proof Overview:

- Let I be as hypothesized. Forcing with $\mathcal{P}(\omega_2)/I$ gives a generic ultrafilter H on ω_2, and also a generic ultrafilter G on ω_1. We get well-founded ultrapowers $M \cong V^{\omega_2}/H$ and $N \cong V^{\omega_1}/G$.
- M, N contain all the reals of $V[H], V[G]$ respectively.

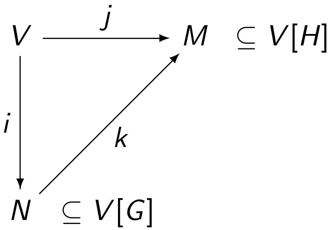
Woodin's Theorem

Theorem (Woodin)

If there is a countably complete, ω_1-dense ideal on ω_2, then CH holds.

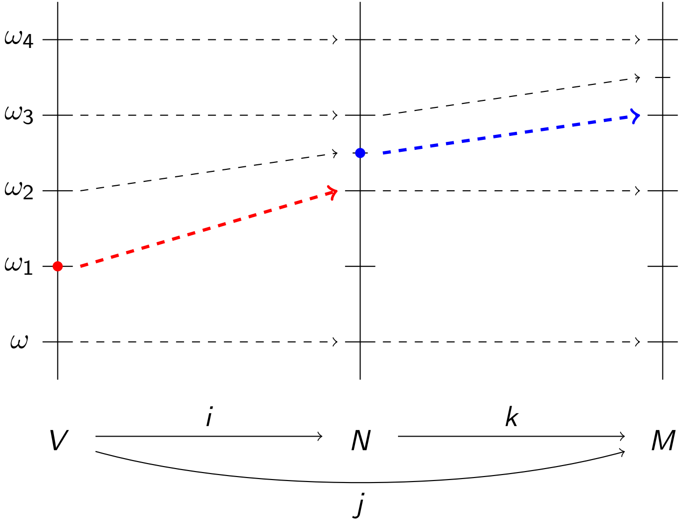
Proof Overview:

- Let I be as hypothesized. Forcing with $\mathcal{P}(\omega_2)/I$ gives a generic ultrafilter H on ω_2, and also a generic ultrafilter G on ω_1. We get well-founded ultrapowers $M \cong V^{\omega_2}/H$ and $N \cong V^{\omega_1}/G$.
- M, N contain all the reals of $V[H], V[G]$ respectively.
- We get a system of embeddings:



Woodin's Theorem

- New reals are added between $V[G]$ and $V[H]$ if and only if CH fails.



Woodin's Theorem

- Since G collapses ω_1, and $\mathcal{P}(\omega_2)/I$ has a a dense subset of size ω_1, the factor forcing to get H from G is at most countable. Therefore, it is equivalent to Cohen forcing if CH fails and trivial if CH holds.

Woodin's Theorem

- Since G collapses ω_1, and $\mathcal{P}(\omega_2)/I$ has a a dense subset of size ω_1, the factor forcing to get H from G is at most countable. Therefore, it is equivalent to Cohen forcing if CH fails and trivial if CH holds.
- Therefore, if CH fails, Cohen forcing is able to create the embedding k. Following a similar strategy to the Gitik-Shelah argument, we ultimately arrive at the same contradiction: two Cohen names for distinct reals, such that their canonical functions agree on a nonmeager set. $\square$

Generalization

The proofs of both theorems use facts specific to ω, $\mathbb{R}$, and Cohen forcing, so it seems impossible to generalize the arguments in a straightforward way. But the following observation turns out to be useful.

Generalization

The proofs of both theorems use facts specific to ω, $\mathbb{R}$, and Cohen forcing, so it seems impossible to generalize the arguments in a straightforward way. But the following observation turns out to be useful.

Preservation Lemma

Suppose κ is a regular cardinal and I is a κ-complete ideal on a set X. Let $\mathbb{P}$ be any notion of forcing, and let G be $\mathbb{P}$-generic. In $V[G]$, let $\bar{I}$ be the ideal generated by I:

$$\bar{I} = \{A \subseteq X : (\exists B \in I) A \subseteq B\}$$

1. If $\mathbb{P}$ is κ-c.c., then $\bar{I}$ is κ-complete in $V[G]$.
2. If $\mathbb{P}$ is κ-c.c. and I is normal, then $\bar{I}$ is normal in $V[G]$.
3. If $|\mathbb{P}| < \kappa$, then for every $\bar{I}$-positive set A, there is an I-positive set $B \in V$ such that $B \subseteq A$.

Generalization

Corollary (Gitik-Shelah)

If I is κ-complete, δ-dense, nowhere-prime ideal, then $\kappa \leq \delta$.

Generalization

Corollary (Gitik-Shelah)

If I is κ-complete, δ-dense, nowhere-prime ideal, then $\kappa \leq \delta$.

Proof: Assume the contrary. Let $\mathbb{P} = Col(\omega, \delta)$. Then in $V^{\mathbb{P}}$, $\bar{I}$ is κ-complete and ω-dense, contradicting the earlier theorem. $\square$

Generalization

Corollary (Gitik-Shelah)

If I is κ-complete, δ-dense, nowhere-prime ideal, then $\kappa \leq \delta$.

Proof: Assume the contrary. Let $\mathbb{P} = Col(\omega, \delta)$. Then in $V^{\mathbb{P}}$, $\bar{I}$ is κ-complete and ω-dense, contradicting the earlier theorem. $\square$

Note: Gitik and Shelah derive this as a corollary to a more general theorem with a much longer proof. The preservation lemma is used in the proof, but they choose not to show how it gives the above short argument.

Generalization

Question (Foreman)

Does the existence of an ω_2-complete, ω_2-dense ideal on ω_3 imply $2^{\omega_1} = \omega_2$?

Generalization

Question (Foreman)

Does the existence of an ω_2-complete, ω_2-dense ideal on ω_3 imply $2^{\omega_1} = \omega_2$?

Answer:

Corollary (E.)

If there is a κ^+-complete, κ^+-dense ideal on κ^{++}, then $2^\kappa = \kappa^+$.

Generalization

Question (Foreman)

Does the existence of an ω_2-complete, ω_2-dense ideal on ω_3 imply $2^{\omega_1} = \omega_2$?

Answer:

Corollary (E.)

If there is a κ^+-complete, κ^+-dense ideal on κ^{++}, then $2^\kappa = \kappa^+$.

Proof: Let I be a κ^+-complete, κ^+-dense ideal on κ^{++}, let $\mathbb{P} = Col(\omega, \kappa)$, and let G be $\mathbb{P}$-generic. Then in $V[G]$, $\kappa^+ = \omega_1$ and $\kappa^{++} = \omega_2$.

Generalization

Question (Foreman)

Does the existence of an ω_2-complete, ω_2-dense ideal on ω_3 imply $2^{\omega_1} = \omega_2$?

Answer:

Corollary (E.)

If there is a κ^+-complete, κ^+-dense ideal on κ^{++}, then $2^\kappa = \kappa^+$.

Proof: Let I be a κ^+-complete, κ^+-dense ideal on κ^{++}, let $\mathbb{P} = Col(\omega, \kappa)$, and let G be $\mathbb{P}$-generic. Then in $V[G]$, $\kappa^+ = \omega_1$ and $\kappa^{++} = \omega_2$.

If there is $f \in V$ which is a surjection from $\mathcal{P}^V(\kappa)$ onto κ^{++}, then this f witnesses the failure of CH in $V[G]$. But by the preservation lemma, $\bar{I}$ is ω_1-complete and ω_1-dense, so by Woodin's theorem, $V[G] \vDash CH$. $\square$

Ulam's Measure Problem

Ulam was the first to show that ω_1 is not a measurable cardinal, i.e. there is no countably complete ideal I on ω_1 such $I \cup I^* = \mathcal{P}(\omega_1)$.

Ulam's Measure Problem

Ulam was the first to show that ω_1 is not a measurable cardinal, i.e. there is no countably complete ideal I on ω_1 such $I \cup I^* = \mathcal{P}(\omega_1)$.

Alaoglu and Erdös extended Ulam's result to show that if $\{I_n : n \in \omega\}$ is a set of countably complete ideals on ω_1, $\bigcup_{n\in\omega} I_n \cup I_n^* \neq \mathcal{P}(\omega_1)$. Ulam asked whether ω_1 many ideals can suffice.

Ulam's Measure Problem

Ulam was the first to show that ω_1 is not a measurable cardinal, i.e. there is no countably complete ideal I on ω_1 such $I \cup I^* = \mathcal{P}(\omega_1)$.

Alaoglu and Erdös extended Ulam's result to show that if $\{I_n : n \in \omega\}$ is a set of countably complete ideals on ω_1, $\bigcup_{n\in\omega} I_n \cup I_n^* \neq \mathcal{P}(\omega_1)$. Ulam asked whether ω_1 many ideals can suffice.

It is easy to see that Ulam's problem has an affirmative solution if there is an ω_1-complete, ω_1-dense ideal on ω_1. What about the other direction?

Ulam's Measure Problem

Ulam was the first to show that ω_1 is not a measurable cardinal, i.e. there is no countably complete ideal I on ω_1 such $I \cup I^* = \mathcal{P}(\omega_1)$.

Alaoglu and Erdös extended Ulam's result to show that if $\{I_n : n \in \omega\}$ is a set of countably complete ideals on ω_1, $\bigcup_{n\in\omega} I_n \cup I_n^* \neq \mathcal{P}(\omega_1)$. Ulam asked whether ω_1 many ideals can suffice.

It is easy to see that Ulam's problem has an affirmative solution if there is an ω_1-complete, ω_1-dense ideal on ω_1. What about the other direction?

Theorem (Taylor)

There is an ω_1-complete, ω_1-dense ideal on ω_1 iff there is a set $\{I_\alpha : \alpha \in \omega_1\}$ of normal ideals on ω_1 such that $\bigcup_{\alpha\in\omega_1} I_\alpha \cup I_\alpha^* = \mathcal{P}(\omega_1)$.

General Measure Problem

Let's use the following abbreviations:

- $d(X, \kappa)$ is the least cardinal δ such that there is a κ-complete, δ-dense, nowhere-prime ideal on X.

General Measure Problem

Let's use the following abbreviations:

- $d(X, \kappa)$ is the least cardinal δ such that there is a κ-complete, δ-dense, nowhere-prime ideal on X.
- $m(X, \kappa)$ is the least cardinal δ such that there is a set $\{I_\alpha : \alpha < \delta\}$ of κ-complete, nowhere-prime ideals on X, where $\mathcal{P}(X) = \bigcup_{\alpha<\delta} I_\alpha \cup I_\alpha^*$.

General Measure Problem

Let's use the following abbreviations:

- $d(X, \kappa)$ is the least cardinal δ such that there is a κ-complete, δ-dense, nowhere-prime ideal on X.
- $m(X, \kappa)$ is the least cardinal δ such that there is a set $\{I_\alpha : \alpha < \delta\}$ of κ-complete, nowhere-prime ideals on X, where $\mathcal{P}(X) = \bigcup_{\alpha<\delta} I_\alpha \cup I_\alpha^*$.
- $nm(X, \kappa)$ means the same except with the requirement that the ideals be normal.

General Measure Problem

Let's use the following abbreviations:

- $d(X, \kappa)$ is the least cardinal δ such that there is a κ-complete, δ-dense, nowhere-prime ideal on X.
- $m(X, \kappa)$ is the least cardinal δ such that there is a set $\{I_\alpha : \alpha < \delta\}$ of κ-complete, nowhere-prime ideals on X, where $\mathcal{P}(X) = \bigcup_{\alpha<\delta} I_\alpha \cup I_\alpha^*$.
- $nm(X, \kappa)$ means the same except with the requirement that the ideals be normal.

Related Results of Taylor

1. $d(\omega_1, \omega_1) = \omega_1 \Leftrightarrow nm(\omega_1, \omega_1) = \omega_1$.
2. $(\forall \kappa) m(\kappa^+, \kappa^+) > \kappa$.
3. $(\forall X) d(X, \omega_1) = \omega \Leftrightarrow m(X, \omega_1) = \omega$.

General Measure Problem

Let's use the following abbreviations:

- $d(X, \kappa)$ is the least cardinal δ such that there is a κ-complete, δ-dense, nowhere-prime ideal on X.
- $m(X, \kappa)$ is the least cardinal δ such that there is a set $\{I_\alpha : \alpha < \delta\}$ of κ-complete, nowhere-prime ideals on X, where $\mathcal{P}(X) = \bigcup_{\alpha<\delta} I_\alpha \cup I_\alpha^*$.
- $nm(X, \kappa)$ means the same except with the requirement that the ideals be normal.

Related Results of Taylor

1. $d(\omega_1, \omega_1) = \omega_1 \Leftrightarrow nm(\omega_1, \omega_1) = \omega_1$.
2. $(\forall\kappa) m(\kappa^+, \kappa^+) > \kappa$.
3. $(\forall X) d(X, \omega_1) = \omega \Leftrightarrow m(X, \omega_1) = \omega$.

The Gitik-Shelah theorem allows us to strengthen the last one.

General Measure Problem

Related to these ideas is the following:

Definition

An ideal I has the κ-refinement property, $RP(\kappa)$, if every for sequence $\langle A_\alpha : \alpha < \kappa \rangle \subseteq I^+$, there is a sequence $\langle B_\alpha : \alpha < \kappa \rangle \subseteq I^+$ which is pairwise disjoint, and $B_\alpha \subseteq A_\alpha$ for all α.

General Measure Problem

Related to these ideas is the following:

Definition

An ideal I has the κ-refinement property, $RP(\kappa)$, if every for sequence $\langle A_\alpha : \alpha < \kappa \rangle \subseteq I^+$, there is a sequence $\langle B_\alpha : \alpha < \kappa \rangle \subseteq I^+$ which is pairwise disjoint, and $B_\alpha \subseteq A_\alpha$ for all α.

It is easy to see that for all regular κ,
$d(\kappa, \kappa) \leq \kappa \Rightarrow nm(\kappa, \kappa) \leq \kappa \Rightarrow m(\kappa, \kappa) \leq \kappa$. Taylor also proved:

Theorem (Taylor)

$nm(\kappa, \kappa) \leq \kappa \Rightarrow$ there is a normal ideal on κ that has $\neg RP(\kappa)$.

General Measure Problem

By combining his results with the following, he was able to prove his theorem about ω_1:

Theorem (Baumgartner-Hajnal-Máté)

If I is a normal ideal on ω_1 which is nowhere ω_1-dense, then I has $RP(\omega_1)$.

General Measure Problem

By combining his results with the following, he was able to prove his theorem about ω_1:

Theorem (Baumgartner-Hajnal-Máté)

If I is a normal ideal on ω_1 which is nowhere ω_1-dense, then I has $RP(\omega_1)$.

The proof of this theorem is specific to ω_1. It takes an $\omega \times \omega_1$ Ulam matrix and, using the assumption of nowhere ω_1-density, refines the matrix one row at a time to a pairwise disjoint matrix. It is not clear how the proof could be modified to go through limit stages.

General Measure Problem

By combining his results with the following, he was able to prove his theorem about ω_1:

Theorem (Baumgartner-Hajnal-Máté)

If I is a normal ideal on ω_1 which is nowhere ω_1-dense, then I has $RP(\omega_1)$.

The proof of this theorem is specific to ω_1. It takes an $\omega \times \omega_1$ Ulam matrix and, using the assumption of nowhere ω_1-density, refines the matrix one row at a time to a pairwise disjoint matrix. It is not clear how the proof could be modified to go through limit stages.

However, using our collapse technique, we can get the following:

Theorem (E.)

Suppose $2^\kappa = \kappa^+$. Then $d(\kappa^+, \kappa^+) = \kappa^+ \Leftrightarrow nm(\kappa^+, \kappa^+) = \kappa^+$.

General Measure Problem

Lemma (E.)

Suppose $2^\kappa = \kappa^+$*. If there is a set* $\{I_\alpha : \alpha < \kappa^+\}$ *of normal,* κ^{++}*-saturated ideals on* κ^+ *such that* $\mathcal{P}(\kappa^+) = \bigcup_{\alpha<\kappa^+} I_\alpha \cup I_\alpha^*$*, then* $d(\kappa^+, \kappa^+) = \kappa^+$*.*

General Measure Problem

Lemma (E.)

Suppose $2^\kappa = \kappa^+$*. If there is a set* $\{I_\alpha : \alpha < \kappa^+\}$ *of normal,* κ^{++}*-saturated ideals on* κ^+ *such that* $\mathcal{P}(\kappa^+) = \bigcup_{\alpha<\kappa^+} I_\alpha \cup I^*_\alpha$*, then* $d(\kappa^+, \kappa^+) = \kappa^+$*.*

Proof Overview:

- If such a collection of ideals exists, Taylor's theorem gives a normal ideal I on κ^+ for which $RP(\kappa^+)$ fails. Call a counterexample to the refinement property an "unrefinable sequence."

General Measure Problem

Lemma (E.)

Suppose $2^\kappa = \kappa^+$. If there is a set $\{I_\alpha : \alpha < \kappa^+\}$ of normal, κ^{++}-saturated ideals on κ^+ such that $\mathcal{P}(\kappa^+) = \bigcup_{\alpha<\kappa^+} I_\alpha \cup I_\alpha^*$, then $d(\kappa^+, \kappa^+) = \kappa^+$.

Proof Overview:

- If such a collection of ideals exists, Taylor's theorem gives a normal ideal I on κ^+ for which $RP(\kappa^+)$ fails. Call a counterexample to the refinement property an "unrefinable sequence."
- Using the cardinal arithmetic assumption, we can build an unrefinable sequence which is guaranteed to remain unrefinable after forcing with $Col(\omega, \kappa)$.

General Measure Problem

Lemma (E.)

Suppose $2^\kappa = \kappa^+$*. If there is a set* $\{I_\alpha : \alpha < \kappa^+\}$ *of normal,* κ^{++}*-saturated ideals on* κ^+ *such that* $\mathcal{P}(\kappa^+) = \bigcup_{\alpha<\kappa^+} I_\alpha \cup I_\alpha^*$*, then* $d(\kappa^+, \kappa^+) = \kappa^+$*.*

Proof Overview:

- If such a collection of ideals exists, Taylor's theorem gives a normal ideal I on κ^+ for which $RP(\kappa^+)$ fails. Call a counterexample to the refinement property an "unrefinable sequence."
- Using the cardinal arithmetic assumption, we can build an unrefinable sequence which is guaranteed to remain unrefinable after forcing with $Col(\omega, \kappa)$.
- Use the BHM theorem in the extension to get a set $A \in V$ such that $\bar{I} \restriction A$ is ω_1-dense in $V[G]$. This property can be pulled back to V. □

General Measure Problem

Now for the main theorem, assume $2^\kappa = \kappa^+$ and $d(\kappa^+, \kappa^+) > \kappa^+$. Let $\{I_\alpha : \alpha < \kappa^+\}$ be any set of normal ideals on κ^+. Let $S = \{\alpha : I_\alpha$ is nowhere κ^{++}-saturated$\}$. For each $\alpha \notin S$, let A_α be such that $I_\alpha \restriction A_\alpha$ is κ^{++}-saturated.

General Measure Problem

Now for the main theorem, assume $2^\kappa = \kappa^+$ and $d(\kappa^+, \kappa^+) > \kappa^+$. Let $\{I_\alpha : \alpha < \kappa^+\}$ be any set of normal ideals on κ^+. Let $S = \{\alpha : I_\alpha$ is nowhere κ^{++}-saturated$\}$. For each $\alpha \notin S$, let A_α be such that $I_\alpha \restriction A_\alpha$ is κ^{++}-saturated.

Using another lemma of Taylor, we are able to find three sets X_0, X_1, Y which are pairwise disjoint, I_α-positive for all $\alpha \in S$, and such that Y is $(I_\beta \restriction A_\beta)$-positive for all $\beta \notin S$.

General Measure Problem

Now for the main theorem, assume $2^\kappa = \kappa^+$ and $d(\kappa^+, \kappa^+) > \kappa^+$. Let $\{I_\alpha : \alpha < \kappa^+\}$ be any set of normal ideals on κ^+. Let $S = \{\alpha : I_\alpha$ is nowhere κ^{++}-saturated$\}$. For each $\alpha \notin S$, let A_α be such that $I_\alpha \restriction A_\alpha$ is κ^{++}-saturated.

Using another lemma of Taylor, we are able to find three sets X_0, X_1, Y which are pairwise disjoint, I_α-positive for all $\alpha \in S$, and such that Y is $(I_\beta \restriction A_\beta)$-positive for all $\beta \notin S$.

By the assumption, my lemma implies that Y can be split into disjoint Y_0, Y_1 which are both $(I_\beta \restriction A_\beta)$-positive for every $\beta \notin S$.

General Measure Problem

Now for the main theorem, assume $2^\kappa = \kappa^+$ and $d(\kappa^+, \kappa^+) > \kappa^+$. Let $\{I_\alpha : \alpha < \kappa^+\}$ be any set of normal ideals on κ^+. Let $S = \{\alpha : I_\alpha$ is nowhere κ^{++}-saturated$\}$. For each $\alpha \notin S$, let A_α be such that $I_\alpha \restriction A_\alpha$ is κ^{++}-saturated.

Using another lemma of Taylor, we are able to find three sets X_0, X_1, Y which are pairwise disjoint, I_α-positive for all $\alpha \in S$, and such that Y is $(I_\beta \restriction A_\beta)$-positive for all $\beta \notin S$.

By the assumption, my lemma implies that Y can be split into disjoint Y_0, Y_1 which are both $(I_\beta \restriction A_\beta)$-positive for every $\beta \notin S$.

Then the sets $X_0 \cup Y_0$, $X_1 \cup Y_1$ witness that $\bigcup_{\alpha<\kappa^+} I_\alpha \cup I_\alpha^* \neq \mathcal{P}(\kappa^+)$. □

Further Directions

Another interesting theorem that can be shown with these techniques is:

Theorem (E.)

Suppose λ is a singular strong limit cardinal, and $2^\lambda < 2^{\lambda^+}$. Then $d(\lambda^+, \lambda^+) > \lambda^+$.

Further Directions

Another interesting theorem that can be shown with these techniques is:

Theorem (E.)

Suppose λ is a singular strong limit cardinal, and $2^\lambda < 2^{\lambda^+}$. Then $d(\lambda^+, \lambda^+) > \lambda^+$.

A direction for further research is to elaborate on the relationships between $d(X, \kappa)$, $m(X, \kappa)$, and cardinal arithmetic. It will be interesting to see what else can be established in ZFC, as well as showing what is independent.

Further Directions

Another interesting theorem that can be shown with these techniques is:

Theorem (E.)

Suppose λ is a singular strong limit cardinal, and $2^\lambda < 2^{\lambda^+}$. Then $d(\lambda^+, \lambda^+) > \lambda^+$.

A direction for further research is to elaborate on the relationships between $d(X, \kappa)$, $m(X, \kappa)$, and cardinal arithmetic. It will be interesting to see what else can be established in ZFC, as well as showing what is independent.

Thank You

References

Baumgartner, J. E.; Hajal, A.; Mate, A. (1973)
Weak saturation properties of ideals.
Infinite and finite sets Vol. I, pp. 137 – 158.

Foreman, Matthew (2010)
Ideals and generic elementary embeddings.
Handbook of set theory. Springer, Dordrecht, 2010, pp. 885–1147.

Gitik, Moti; Shelah, Saharon (1989)
Forcings with ideals and simple forcing notions.
Israel Journal of Mathematics 68(1989), no. 2, 129 – 160.

Taylor, Alan (1979)
Regularity properties of ideals and ultrafilters.
Annals of Mathematical Logic 16(1979), 33 – 55.

Taylor, Alan (1980)
On saturated sets of ideals and Ulam's problem.
Fundamenta Mathematicae 109 (1980), no. 1, 37 – 53.