Light Logics for Polynomial Time Computations

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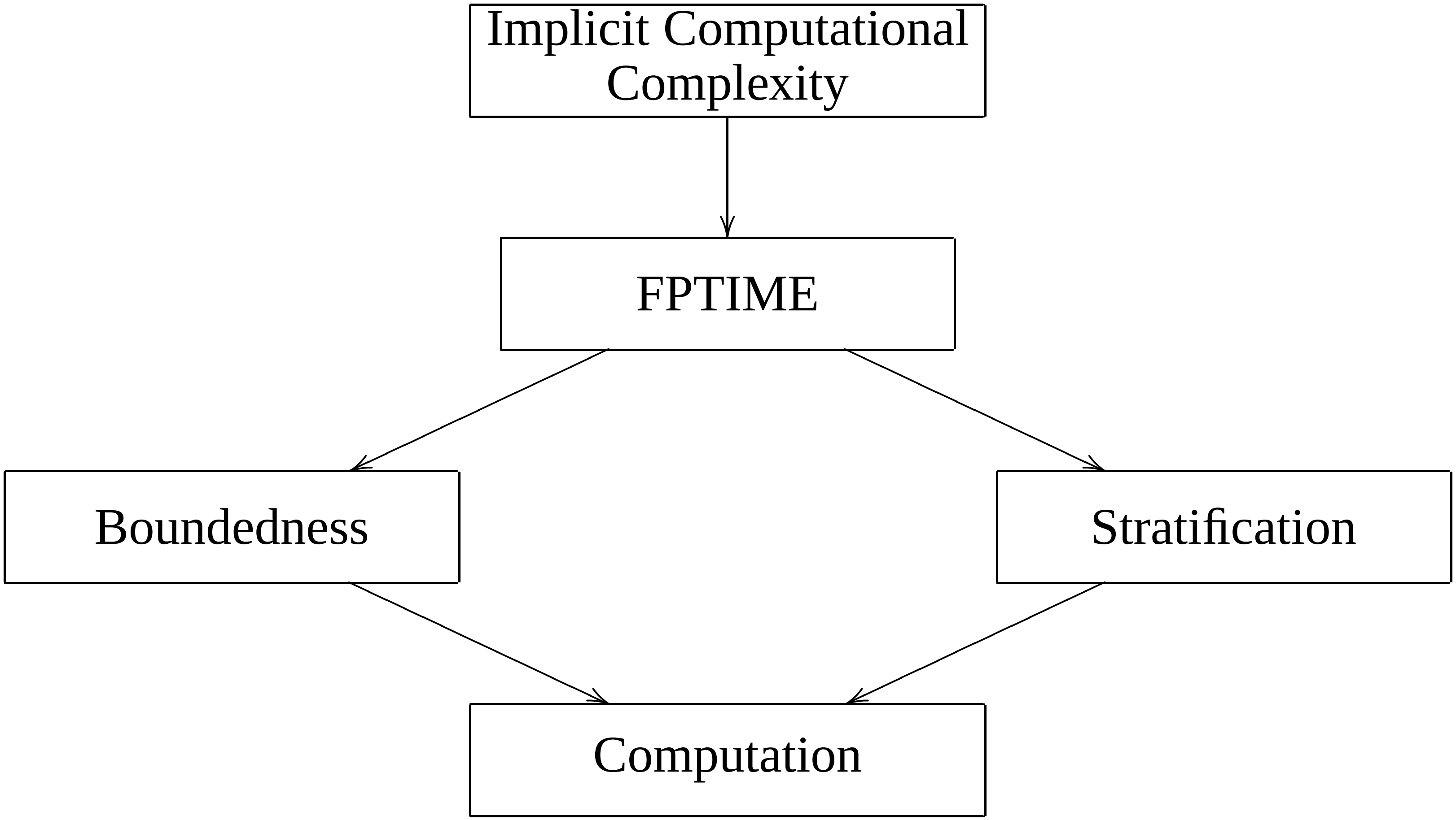
Outline



Implicit Computational Complexity
FPTIME
Boundedness
Stratification
Computation

Implicit Computational Complexity

Implicit Computational Complexity: Motivations

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- It generally borrows techniques from Mathematical Logic :
  - Recursion Theory, FPTIME = Predicative Recursion on Notation
  - Structural Proof Theory, FPTIME = Bounded/Light Linear Logic
  - Model Theory, FPTIME = PR Functions over Finite Structures

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  - Model Theory, FPTIME = PR Functions over Finite Structures
- Our approach:

Characterizing the class FPTIME

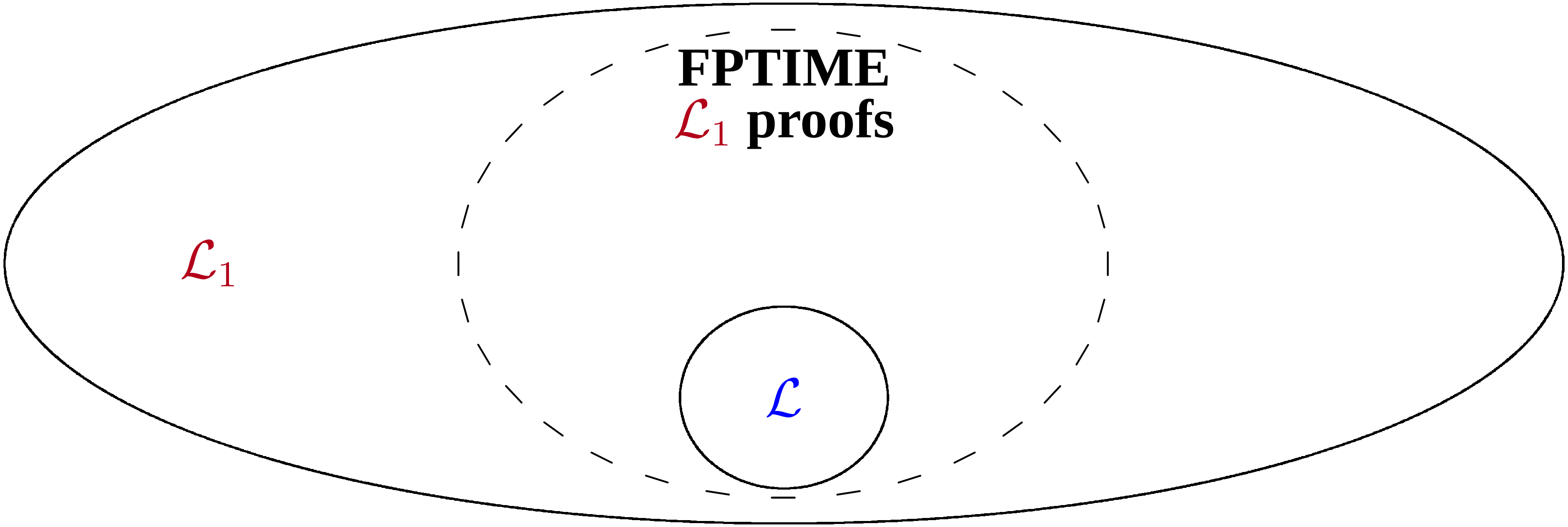
FPTIME is the class of functions (and not only decision problems) computable in polynomial time by a deterministic Turing Machine.

A Logic $\mathcal{L}$ characterizes FPTIME if:

- **Soundness**: Every proof/program in $\mathcal{L}$ can be evaluated in polynomial time.
- **Extensional Completeness**: Every TM computing a function in FPTIME can be simulated by means of a proof/program in $\mathcal{L}$.

Characterizing the class FPTIME

Extensional Completeness does not say anything about the expressivity of $\mathcal{L}$. For instance:



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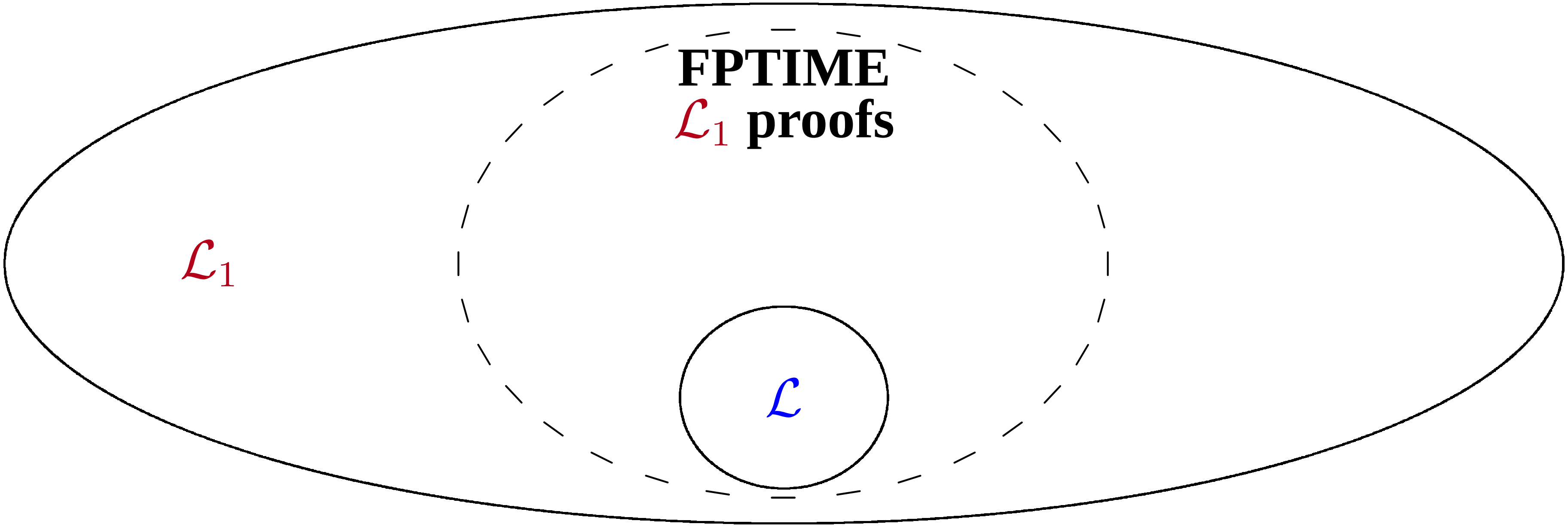

Intensional Completeness:
The Logic $\mathcal{L}$ captures all the FPTIME proofs of $\mathcal{L}_1$.

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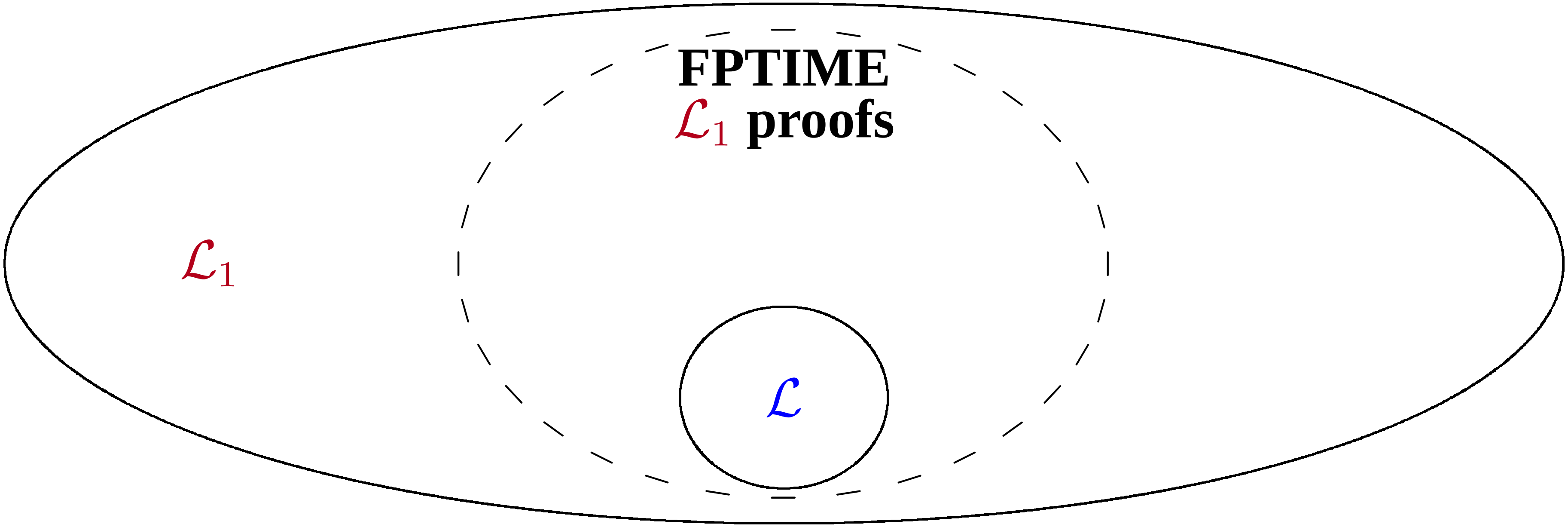

Intensional Completeness: (Usually undecidable!)
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- Intuitively, FPTIME is the class of programs obtained by permitting arbitrary compositions of polynomial iterations.

$$r(x) \circ q(x) = p(x) \qquad \underbrace{p(x) \circ p(x) \circ \cdot \circ p(x)}_{x\text{-times}} = e(x)$$

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- **Recurrent idea:** Allows definitions by iteration of polynomials but forbids dangerous iterations of polynomials.

Road-map

System	Soundness	Extensional Completeness	Intensional Completeness	Decidability
Bounded Recursion on Notation	X	X	X	
Predicative Recursion on Notation	X	X		X
Light Affine Logic	X	X		X
Bounded Linear Logic	X	X	$\approx$	

First approach: Bounded Recursion on notation

The first implicit characterization of FPTIME [Cobham65] uses BRN:

$$\begin{array}{rcl} f(\epsilon, \vec{y}) & = & g(\vec{y}) \\ f(\mathbf{0}x, \vec{y}) & = & h_0(x, \vec{y}, f(x, \vec{y})) \\ f(\mathbf{1}x, \vec{y}) & = & h_1(x, \vec{y}, f(x, \vec{y})) \\ f(x, \vec{y}) & \leq & k(x, \vec{y}) \end{array}$$

with a smash function $x\#y = \underbrace{\mathbf{10}\cdots\mathbf{0}}_{|x|\cdot|y|}$ and few other basic functions.

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with a smash function $x \# y = \underbrace{\mathbf{1}\mathbf{0} \cdots \mathbf{0}}_{|x| \cdot |y|}$ and few other basic functions.

Pros:

- \+ No explicit machine model.
- \+ Very expressive.

Cons:

- \- The bound is not really implicit.
- \- The bound is difficult to check (undecidable!).

Second approach: Predicative Recursion on notation

Another approach is by using PRN [Bellantoni& Cook91,Leivant91]:

$$\begin{array}{rcl} f(\epsilon, \vec{z}\,;\vec{y}) & = & g(\vec{z}\,;\vec{y}) \\ f(\mathbf{0}x, \vec{z}\,;\vec{y}) & = & h_0(x, \vec{z}\,;\vec{y}, f(x, \vec{z}\,;\vec{y})) \\ f(\mathbf{1}x, \vec{z}\,;\vec{y}) & = & h_1(x, \vec{z}\,;\vec{y}, f(x, \vec{z}\,;\vec{y})) \end{array}$$

Every function $f(\vec{x}\,;\vec{y})$ has normal arguments $\vec{x}$ and safe arguments $\vec{y}$.
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Every function $f(\vec{x} \,\mathbf{;}\, \vec{y})$ has normal arguments $\vec{x}$ and safe arguments $\vec{y}$.
Soundness: the result of an iteration cannot be a recurrence argument.

Pros:

- **+** The bound is no more explicit.
- **+** Simple syntactic criterion.

Cons:

- **-** Poor expressivity.
- **-** Inherently first order.

Stratification

Paradox as the worst complexity...

Through the proofs-as-programs correspondence Intuitionistic Logic with type fixpoints corresponds to a system of recursive types.

$$\frac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\ (Ax) \qquad \frac{\Gamma \vdash \mathtt{N} : A \qquad \Delta, \mathtt{x} : A \vdash \mathtt{M} : B}{\Gamma, \Delta \vdash \mathtt{M}[\mathtt{x}/\mathtt{N}]B}\ (cut)$$

$$\frac{\Gamma, \mathtt{x} : A \vdash \mathtt{M} : B \quad \mathtt{x} \notin FV(\Gamma)}{\Gamma \vdash \lambda \mathtt{x}.\mathtt{M} : A \Rightarrow B}\ (\Rightarrow R) \qquad \frac{\Gamma \vdash \mathtt{N} : A \quad \mathtt{x} : B, \Delta \vdash \mathtt{M} : C}{\mathtt{y} : A \Rightarrow B, \Gamma, \Delta \vdash \mathtt{M}[\mathtt{yN}/\mathtt{x}] : C}\ (\Rightarrow L)$$

$$\frac{\Gamma \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} : B \vdash \mathtt{M} : A}\ (w) \qquad \frac{\Gamma, \mathtt{x} : B, \mathtt{x} : B \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} : B \vdash \mathtt{M} : A}\ (c)$$

$$\frac{\Gamma \vdash \mathtt{M} : A \quad A = B}{\Gamma \vdash \mathtt{M} : B}\ (= R) \qquad \frac{\Gamma, \mathtt{x} : A \vdash \mathtt{M} : C \quad A = B}{\Gamma, \mathtt{x} : B \vdash \mathtt{M} : C}\ (= L)$$

Non termination is the worst use of resource

For what follows it is instructive to look how $(\lambda \mathtt{x}.\mathtt{xx})\lambda \mathtt{x}.\mathtt{xx}$ can be typed using the fixpoint $A = A \Rightarrow \bot$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{x} : A \vdash \mathtt{x} : A \quad \mathtt{x} : \bot \vdash \mathtt{x} : \bot}{\mathtt{x} : A, \mathtt{x} : A \Rightarrow \bot \vdash \mathtt{xx} : \bot}\,(\Rightarrow L)}{\mathtt{x} : A, \mathtt{x} : A \vdash \mathtt{xx} : \bot}\,(= L)}{\mathtt{x} : A \vdash \mathtt{xx} : \bot}\,(c)}{\vdash \lambda \mathtt{x}.\mathtt{xx} : A \Rightarrow \bot}\,(\Rightarrow R) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{x} : A \vdash \mathtt{x} : A \quad \mathtt{x} : \bot \vdash \mathtt{x} : \bot}{\mathtt{x} : A, \mathtt{x} : A \Rightarrow \bot \vdash \mathtt{xx} : \bot}\,(\Rightarrow L)}{\mathtt{x} : A, \mathtt{x} : A \vdash \mathtt{xx} : \bot}\,(= L)}{\mathtt{x} : A \vdash \mathtt{xx} : \bot}\,(c)}{\vdash \lambda \mathtt{x}.\mathtt{xx} : A \Rightarrow \bot}\,(\Rightarrow R)}{\vdash \lambda \mathtt{x}.\mathtt{xx} : A}\,(= R)}{\vdash (\lambda \mathtt{x}.\mathtt{xx})\lambda \mathtt{x}.\mathtt{xx} : \bot}\,(cut)$$

Remark: contraction is necessary.

Intuitionistic Linear Logic with fixpoints: $\mathbf{ILL}_\mu$

$$\frac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\ (Ax) \qquad \frac{\Gamma \vdash \mathtt{N} : A \qquad \Delta, \mathtt{x} : A \vdash \mathtt{M} : B}{\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{x}] : B}\ (cut)$$

$$\frac{\Gamma, \mathtt{x} : A \vdash \mathtt{M} : B \quad \mathtt{x} \notin FV(\Gamma)}{\Gamma \vdash \lambda \mathtt{x}.\mathtt{M} : A \multimap B}\ (\multimap R) \quad \frac{\Gamma \vdash \mathtt{N} : A \qquad \mathtt{x} : B, \Delta \vdash \mathtt{M} : C}{\mathtt{y} : A \multimap B, \Gamma, \Delta \vdash \mathtt{M}[\mathtt{yN}/\mathtt{x}] : C}\ (\multimap L)$$

$$\frac{\Gamma \vdash \mathtt{M} : A \quad A = B}{\Gamma \vdash \mathtt{M} : B}\ (= R) \qquad \frac{\Gamma, \mathtt{x} : A \vdash \mathtt{M} : C \quad A = B}{\Gamma, \mathtt{x} : B \vdash \mathtt{M} : C}\ (= L)$$

$$\frac{!\Gamma \vdash \mathtt{M} : A}{!\Gamma \vdash \mathtt{M} : !A}\ (p) \qquad \frac{\Gamma, \mathtt{x} : B \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} : !B \vdash \mathtt{M} : A}\ (d)$$

$$\frac{\Gamma \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} : !B \vdash \mathtt{M} : A}\ (w) \qquad \frac{\Gamma, \mathtt{x} : !B, \mathtt{x} : !B \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} : !B \vdash \mathtt{M} : A}\ (c)$$

Non termination in $\mathbf{ILL}_\mu$

Obviously $(\lambda \mathtt{x}.\mathtt{xx})\lambda \mathtt{x}.\mathtt{xx}$ is typable in ILL_μ by means of the type fixpoint $A =!A \multimap \bot$ (which is the usual translation of $A = A \Rightarrow \bot$).

$$
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\dfrac{
\dfrac{\mathtt{x} :!A \vdash \mathtt{x} :!A \qquad \mathtt{x} : \bot \vdash \mathtt{x} : \bot}{\mathtt{x}!A, \mathtt{x} :!A \multimap \bot \vdash \mathtt{xx} : \bot}\;(\multimap L)
}{\mathtt{x}!A, \mathtt{x} : A \vdash \mathtt{xx} : \bot}\;(= L)
}{\color{red}{\mathtt{x}!A, \mathtt{x} :!A \vdash \mathtt{xx} : \bot}}\;\color{red}{(d)}
}{\mathtt{x} :!A \vdash \mathtt{xx} : \bot}\;(c)
}{\vdash \lambda \mathtt{x}.\mathtt{xx} :!A \multimap \bot}\;(\multimap R)
\qquad
\dfrac{
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\dfrac{\mathtt{x} :!A \vdash \mathtt{x} :!A \qquad \mathtt{x} : \bot \vdash \mathtt{x} : \bot}{\mathtt{x}_1!A, \mathtt{x}_2 :!A \multimap \bot \vdash \mathtt{xx} : \bot}\;(\multimap L)
}{\mathtt{x}!A, \mathtt{x} : A \vdash \mathtt{xx} : \bot}\;(= L)
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}{\mathtt{x} :!A \vdash \mathtt{xx} : \bot}\;(c)
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}{\color{blue}{\vdash \lambda \mathtt{x}.\mathtt{xx} :!A}}\;\color{blue}{(p)}
}{\vdash (\lambda \mathtt{x}.\mathtt{xx})\lambda \mathtt{x}.\mathtt{xx} : \bot}\;(cut)
$$

Remark: contraction, promotion and dereliction are necessary.

Towards Light Logics

The idea of Light Logics is to limit the power of the structural rules.

- First idea: forbid contraction.
  - \+ Consistent Set theory with a full comprehension scheme.
  - \- Too weak for polynomial time.

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The idea of Light Logics is to limit the power of the structural rules.

- □ First idea: forbid contraction.
  - \+ Consistent Set theory with a full comprehension scheme.
  - \- Too weak for polynomial time.
- □ Second idea: control the other structural rules:
  - i) Proof Stratification.
  - ii) Avoiding the monoidalness of !.

Stratification

We can think to the promotion rule as a **box**:

$$\boxed{\dfrac{\dfrac{\vdots}{!\Gamma \vdash \mathtt{M} : A}}{!\Gamma \vdash \mathtt{M} : !A}}$$

and we say **depth** of a rule, the number of boxes containing it.

Stratification

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and we say **depth** of a rule, the number of boxes containing it.

Non-stratification of a proof in Linear Logic.

The depth of any rule **can** change during the cut elimination.

$$!A \multimap !!A \qquad !A \multimap A \qquad !A \multimap !A \otimes !A \qquad !A_1 \otimes !A_2 \multimap !A$$

Stratification

If we remove the principles:

$$!A \multimap !!A \qquad !A \multimap A$$

we have instead:

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Stratification: The depth of any rule **cannot** change during the cut elimination.

So, we can consider only the rules

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\ (\Box) \qquad \frac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ (\nabla)$$

Reduction at depth i

A cut elimination step at depth i:

i) duplicate part of the proof at depth $i+1$,
ii) decrease the number of rules at depth i,
iii) does not affect the depths $j < i$,
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This can be iterated to obtain an exponential at each fixed depth i.

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$$\dfrac{\dfrac{\Gamma, A \vdash C}{\Gamma \vdash A \multimap C} \quad \dfrac{\Delta \vdash A \quad C, \Sigma \vdash B}{\Delta, A \multimap C, \Sigma \vdash B}}{!\Gamma, \Delta, \Sigma \vdash B}\ (cut) \mapsto \dfrac{\dfrac{\Delta \vdash A \quad \Gamma, A \vdash C}{\Delta, \Gamma \vdash C}\ (cut) \quad C, \Sigma \vdash B}{\Delta, \Gamma, \Sigma \vdash B}\ (cut)$$

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Because of the **stratification!**

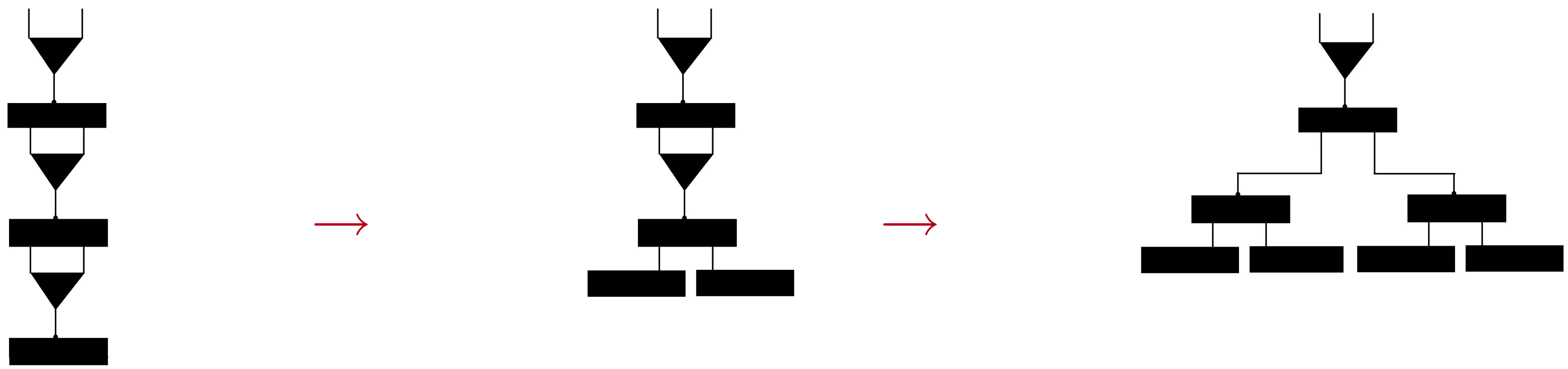
Stratified Reduction

We can perform a **depth-by-depth** reduction:

$$
\left\|\begin{array}{l} |\Pi|_0 \\ |\Pi|_1 \\ |\Pi|_2 \\ |\Pi|_3 \\ \vdots \\ |\Pi|_d \end{array}\right\|
\to_0^*
\left\|\begin{array}{l} |\Pi|_0 \\ 2^{|\Pi|_1} \\ 2^{|\Pi|_2} \\ 2^{|\Pi|_3} \\ \\ 2^{|\Pi|_d} \end{array}\right\|
\to_1^*
\left\|\begin{array}{l} |\Pi|_0 \\ 2^{|\Pi|_1} \\ 2^{2^{|\Pi|_2}} \\ 2^{2^{|\Pi|_3}} \\ \\ 2^{2^{|\Pi|_d}} \end{array}\right\|
\to_2^*
\left\|\begin{array}{l} |\Pi|_0 \\ 2^{|\Pi|_1} \\ 2^{2^{|\Pi|_2}} \\ 2^{2^{2^{|\Pi|_3}}} \\ \\ 2^{2^{2^{|\Pi|_d}}} \end{array}\right\|
\ldots
\left\|\begin{array}{l} |\Pi|_0 \\ 2^{|\Pi|_1} \\ 2^{2^{|\Pi|_2}} \\ 2^{2^{2^{|\Pi|_3}}} \\ \vdots \\ 2^{\cdot^{\cdot^{\cdot^{2^{2^{|\Pi|_d}}}}}} \end{array}\right\|
$$

The exponential blow-up

Most complex element **Most complex element reduced**



- The duplication of boxes depending over (more than one) free variables allows exponential time normalization.
- Limiting this kind of behavior corresponds to reduce the complexity of the normalization to polynomial time.

Light Affine Logic

$$\frac{\Gamma \vdash A \quad \Gamma \subseteq \{B\}}{!\Gamma \vdash !A}\ (p) \qquad \frac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ (c) \qquad \frac{\Gamma, \Delta \vdash A}{\S\Gamma, !\Delta \vdash \S A}\ (d)$$

Light Affine Logic

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The restriction on the (p) rule corresponds to rule out the law:

$$!A_1 \otimes \ldots \otimes !A_n \multimap !A$$

The rules (d) corresponds to re-introduce a weak version of it:

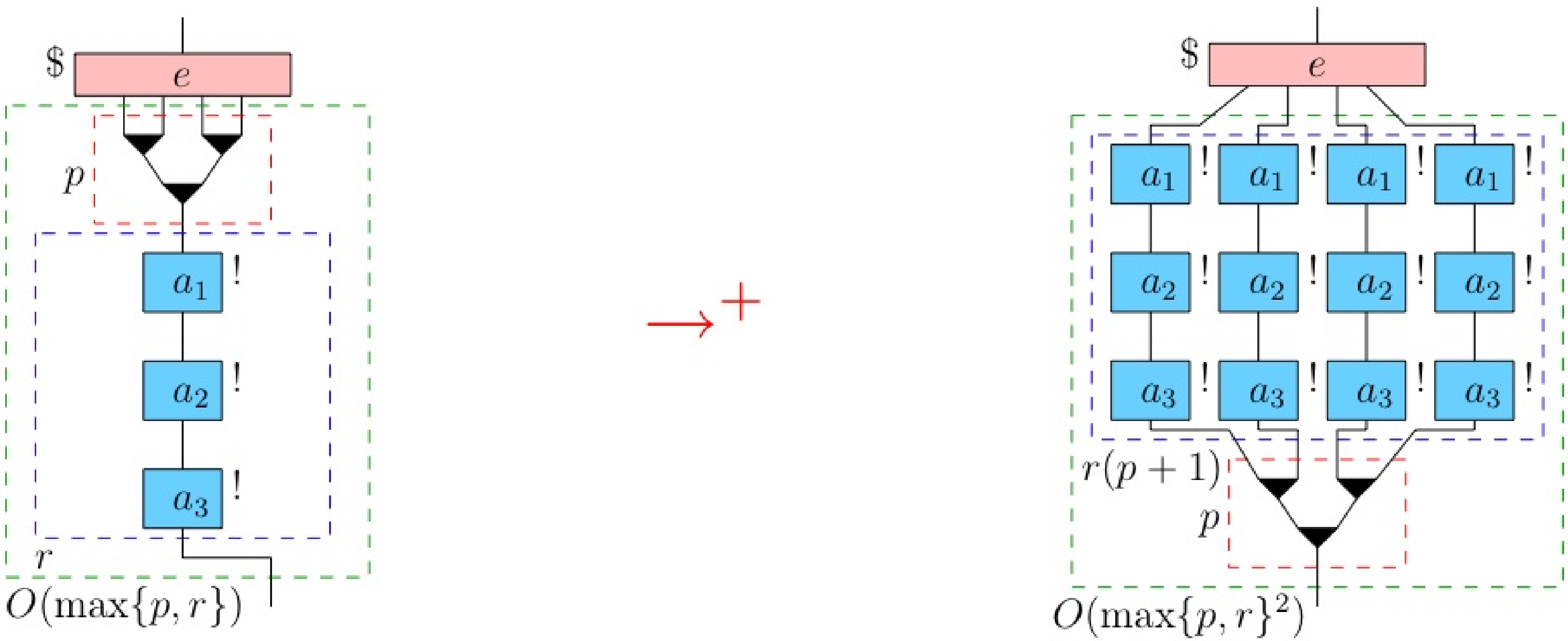
$$!A_1 \otimes \ldots \otimes !A_n \multimap \S A$$

Light Affine Logic

$$\boxed{\dfrac{\Gamma \vdash A \quad \Gamma \subseteq \{B\}}{!\Gamma \vdash !A}\ (p)} \qquad \dfrac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ (c) \qquad \boxed{\dfrac{\Gamma, \Delta \vdash A}{\S\Gamma, !\Delta \vdash \S A}\ (d)}$$

- !-boxes ((p) rules) can be duplicated.
- §-boxes ((d) rules) cannot be duplicated

A sketch of LAL soundness - 1



The size $|\Pi|_j$ of depth $i < j$ becomes (at most) $|\Pi|_j^2$ at each round i.

A sketch of LAL soundness - 2

We can perform a **depth-by-depth** reduction:

$$\left\| \begin{array}{l} |\Pi|_0 \\ |\Pi|_1 \\ |\Pi|_2 \\ |\Pi|_3 \\ \vdots \\ |\Pi|_d \end{array} \right\| \rightarrow_0^* \left\| \begin{array}{l} |\Pi|_0 \\ (|\Pi|_1)^2 \\ (|\Pi|_2)^2 \\ (|\Pi|_3)^2 \\ \\ (|\Pi|_d)^2 \end{array} \right\| \rightarrow_1^* \left\| \begin{array}{l} |\Pi|_0 \\ (|\Pi|_1)^2 \\ ((|\Pi|_2)^2)^2 \\ ((|\Pi|_3)^2)^2 \\ \\ ((|\Pi|_d)^2)^2 \end{array} \right\| \rightarrow_2^* \left\| \begin{array}{l} |\Pi|_0 \\ (|\Pi|_1)^2 \\ ((|\Pi|_2)^2)^2 \\ (((|\Pi|_3)^2)^2)^2 \\ \\ (((|\Pi|_d)^2)^2)^2 \end{array} \right\| \cdots \left\| \begin{array}{l} |\Pi|_0 \\ (|\Pi|_1)^2 \\ ((|\Pi|_2)^2)^2 \\ (((|\Pi|_3)^2)^2)^2 \\ \vdots \\ (((|\Pi|_d)^2)^2 \cdots)^2 \end{array} \right\|$$

A sketch of LAL soundness - 3

The size of the proof Π after the stratified reduction is bounded by:

$$O\left(|\Pi|^{2^d}\right)$$

- d is the maximal depth of a rule in Π.

The same method can be used to reason about reduction steps, hence this gives a bound on the number of β-normalization steps.

Note: in LAL, data types have a fixed depth. This means that the depth depends just on the program part, hence we can work in polynomial time.

A sketch of LAL Completeness

- We have addition and multiplication in LAL as:

$$\mathbf{add} : \mathbf{N} \multimap \mathbf{N} \multimap \mathbf{N} \qquad \mathbf{mult} : \mathbf{N} \multimap \mathbf{N} \multimap \S\mathbf{N}$$

So, we can program a polynomial $p(x)$ of degree d as:

$$\mathbf{p} : \mathbf{N} \multimap \S^{2d}\mathbf{N}$$

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- Transition on Turing Machines (TM) can be programmed as:

$$\mathbf{tr} : \mathbf{TM} \multimap \mathbf{TM}$$

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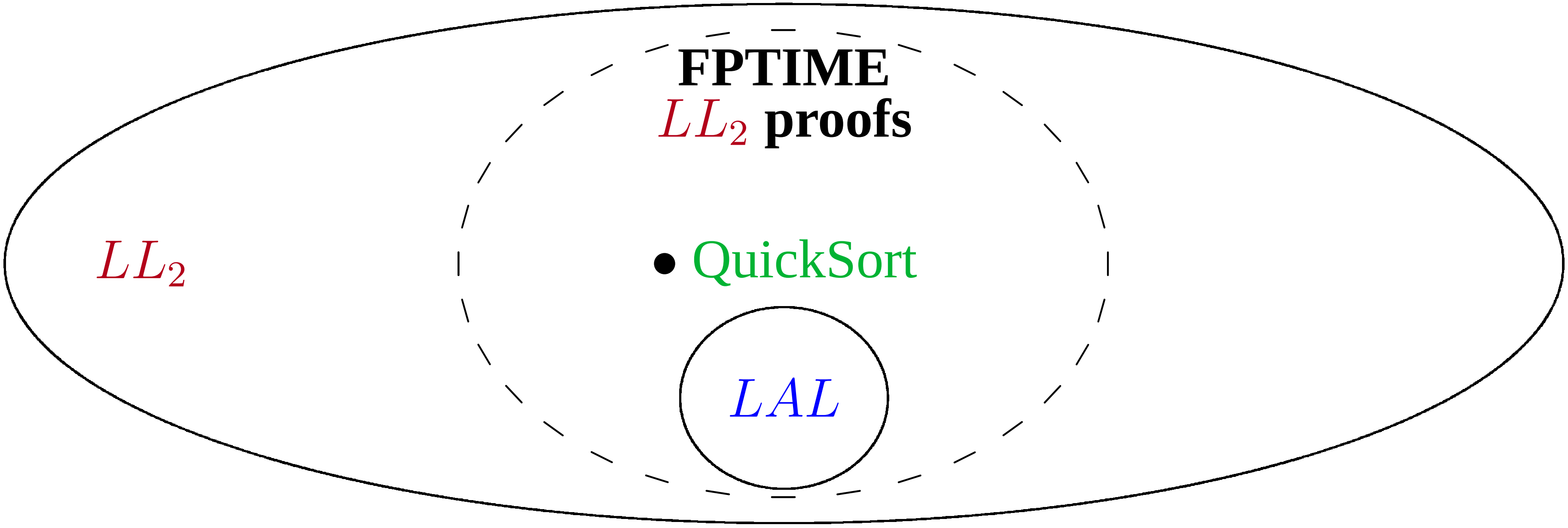
- Transition on Turing Machines (TM) can be programmed as:

$$\mathbf{tr} : \mathbf{TM} \multimap \mathbf{TM}$$

- By some manipulations we can obtain FPTIME Turing machines as:

$$\mathbf{t} : \mathbf{W} \multimap \S^{2d+1}\mathbf{W}$$

Lack of Expressivity



Back to Boundedness

Bounded Linear Logic - 1

One of the earliest examples of a Linear Logic system capturing FPTIME.
Formulas:

$$A ::= \alpha(p_1, \ldots, p_n) \mid A \otimes A \mid A \multimap A \mid \forall \alpha . A \mid \; !_{x<p} A$$

The idea is very simple:

$$!_{x<5} A = A[0/x] \otimes \cdots \otimes A[4/x]$$

However, the technicalities are increased by the fact that p is in general a polynomial expression, e.g.

$$!_{x<y^2} A = A[0/x] \otimes \cdots \otimes A[y^2 - 1/x]$$

This is however also it strength.

Bounded Linear Logic - 2

Rules:

$$\frac{\Gamma \vdash B}{\Gamma, !_{x<w} A \vdash B}\ (w) \qquad \frac{\Gamma, A[x := 0] \vdash B}{\Gamma, !_{x<1+w} A \vdash B}\ (d)$$

$$\frac{\Gamma, !_{x<p} A, !_{x<q} A[x := p + x] \vdash B}{\Gamma, !_{x<p+q+w} A \vdash B}\ (c)$$

$$\frac{!_{z<q_1(x)} A_1[y := v_1(x) + z], \ldots, !_{z<q_n(x)} A_n[y := v_n(x) + z] \vdash B}{!_{y<v_i(p)+w_1} A_1, \ldots, !_{y<v_n(p)+w_n} A_n \vdash !_{x<p} B}\ (p)$$

where $v_i(x) = \sum_{z<x} q_i(z)$.

A source of inspiration: Bounded Linear Logic - 2

Bounded Linear Logic is sound and complete for FPTIME.

Moreover it has some interesting properties.

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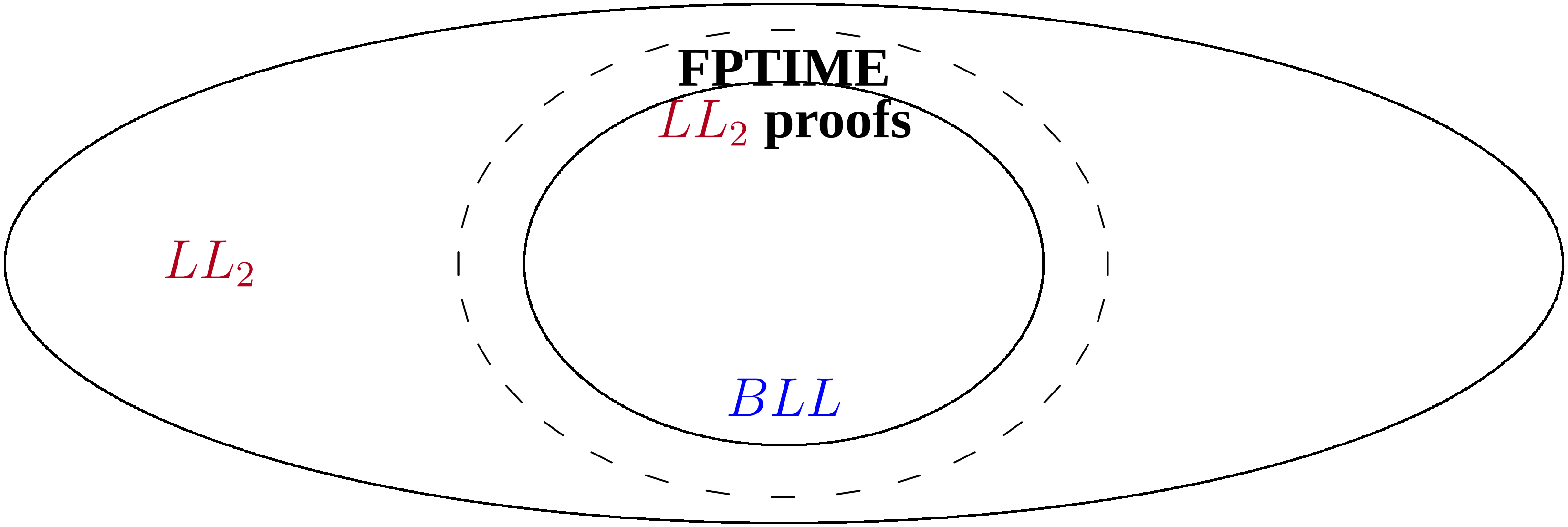


A source of inspiration: Bounded Linear Logic - 2

Bounded Linear Logic is sound and complete for FPTIME.

Moreover it has some interesting properties.

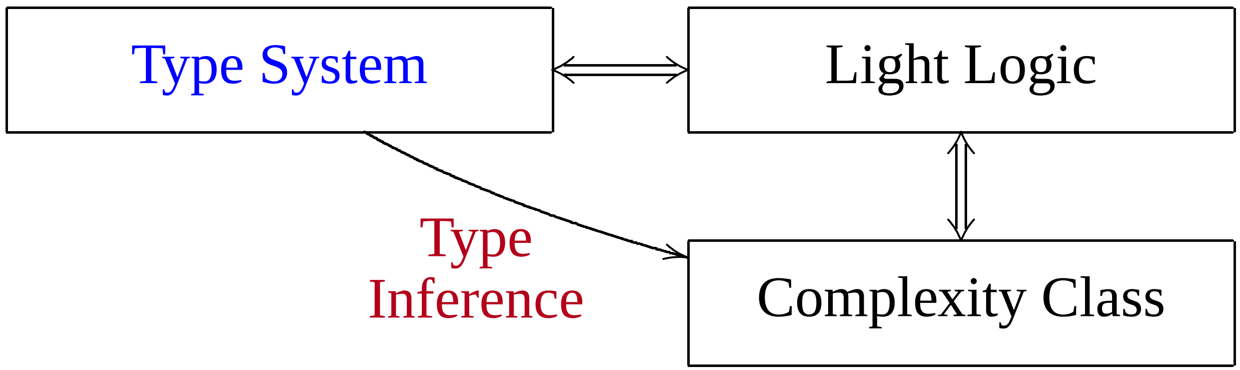
- The "bounding" part can be seen as a kind of extra-conditions. As an example:

$$\frac{\Gamma, !_{x<p}A, !_{x<q}A[x := p + x] \vdash B \quad p + q \leq r}{\Gamma, !_{x<r}A \vdash B}\ (c)$$

Computation

Light Type Systems

Stratified Light Logics have been used to design type systems:



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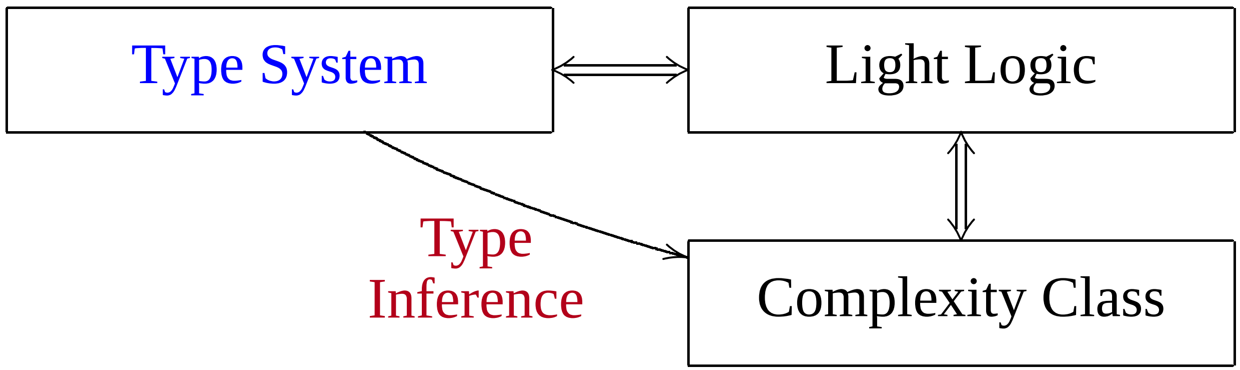
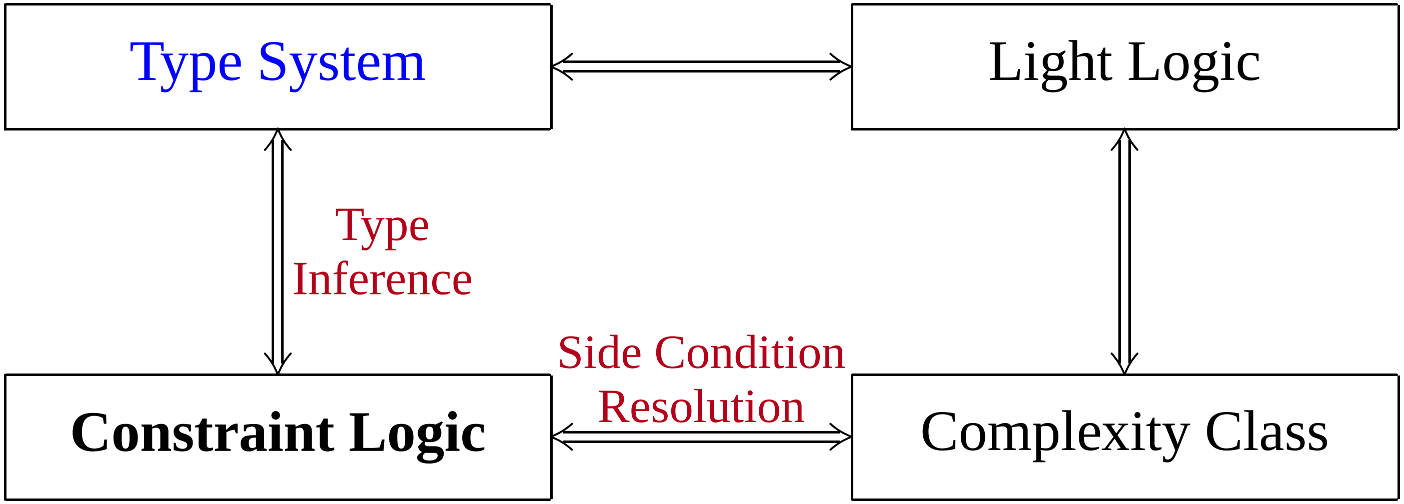


This approach is effective, flexible and robust:

- □ Type Inference usually decidable in polynomial time.
- □ Useful to characterize different complexity classes: FPTIME, ELEMENTARY, FPSPACE, FNPTIME, FEXPTIME.
- □ With minor modifications they are useful for: higher-order, different recursion schemes, control operators, multithreding and side effects.

Relative Complete Type Systems

Boundedness can be used in a slightly different way.



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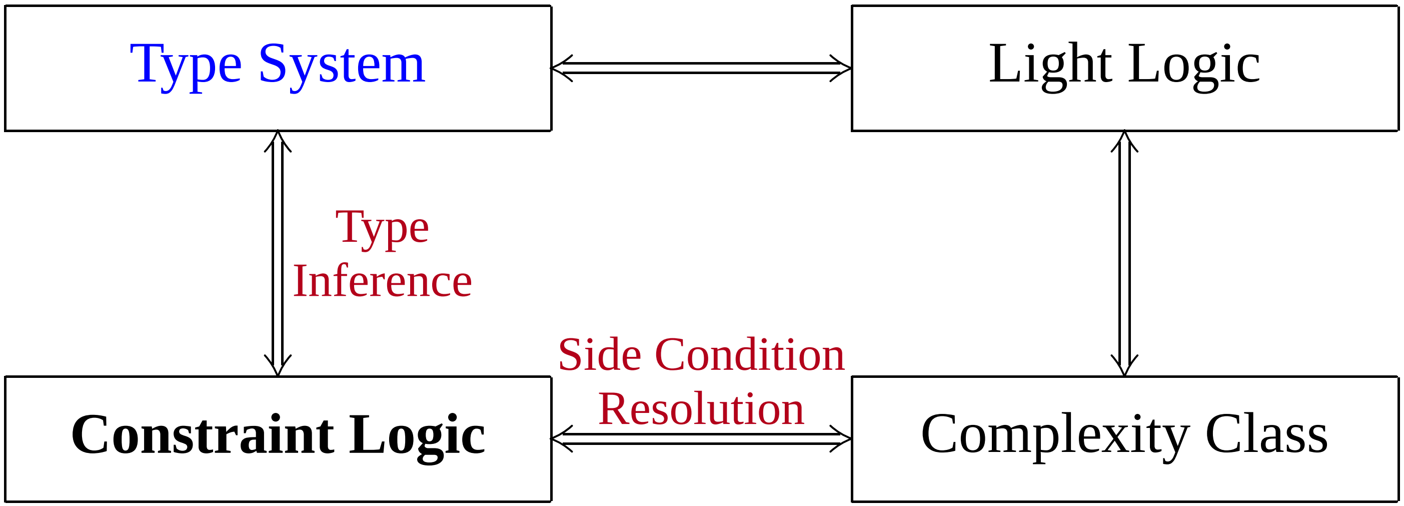

- This approach can give a system that is relatively complete with respect to the solvability of the side conditions by an oracle.
- This gives a general method to analyze program complexity.
- A similar approach has been used in an interactive framework for the class LOGSPACE.

THANKS