Orbits of D-maximal sets in $\mathscr{E}$

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Outline

1. Background
2. Automorphisms
   - Basic Work
   - Advanced Automorphism Methods
3. $\mathscr{D}$-maximal sets

Notation

- ω is the natural numbers. $\overline{X} = w - X$
- $p[X]$ denotes the image of X under p.
- W_e is the domain of the e-th Turing machine
- A_s is the set of elements enumerated into A by stage s.
- All sets are c.e. unless otherwise noted. R_i is assumed to be computable

Lattice of C.E. Sets

Definition (Lattice of c.e. sets)

- $\langle \{W_e, e \in \omega\} . \subseteq \rangle = \mathscr{E}$ are the c.e. sets under inclusion.
- $\mathscr{E}^*$ is $\mathscr{E}$ modulo the ideal $\mathscr{F}$ of finite sets.

Question (Motivating Questions)

- What are the automorphisms of $\mathscr{E}$? $\mathscr{E}^*$?
- What are the definable classes? Orbits?

Simple Automorphisms

- Permutations p of ω induce maps $\mathbf{\Upsilon}(A) = p[A]$ respecting $\subseteq$.
- Any permutation taking c.e. sets to c.e. sets is automatically an automorphism.
- Computable permutations (aka recursive isomorphisms) induce (ω many) automorphisms.

Theorem

All creative sets belong to the same orbit.

Proof.

It is well known that the creative sets are recursively isomorphic. □

How Many Automorphisms?

Theorem (Lachlan)

There are 2^ω *automorphisms of* $\mathscr{E}^*$ *(and* $\mathscr{E}$*)*

Idea

- Build permutations as limit of computable permutations $p_f = \bigcup_{\sigma \in 2^{<\omega}} p_\sigma$ (Respects $\subseteq$).
- Ensure that $\Upsilon(W_e) = R \cup p_\sigma[W_{e_1}] \cup p_\sigma[W_{e_2}]$ where $W_e = R \cup W_{e_1} \cup W_{e_2}$. (Ensures images are c.e.).
- Build so if $\sigma \mid \tau$ then for some A, $p_\sigma(A) \neq^* p_\tau(A)$.

Building Continuum Many Automorphisms

Idea

Build $R_0 \supset R_1 \supset \ldots$ with members of R_e sharing the same e-state and leaving us free to define permutation on R_e as we wish. But first we see we have two choices for this permutation in non-trivial cases.

Lemma

If $R \supset_\infty R \cap W \supset_\infty \emptyset$ then there are computable permutations p_0, p_1 of R with $p_0[W \bigcap R] \neq^* p_1[W \bigcap R]$.

Proof.

Let $S \subset W \bigcap R$ infinite computable subset. Pick p_0 to be the identity and p_1 to exchange S and $R - S$, i.e., p_1 moves infinitely many points outside of W into W. □

Glueing Permutations

Construction

Assume R_n, p_σ are defined. $(R_0 = \omega, p_\lambda = \emptyset)$

1. If W_n almost avoids or contains R_n finitely modify R_{n+1}, p_σ to eliminate the exceptions.
2. Otherwise let $R_{n+1} \subset_\infty W_n \cap R_n$. $W_n, R_n - R_{n+1}$ satisfy the lemma.
    - For each maximal σ with p_σ defined let $p_{\sigma\hat{}\langle\langle j\rangle\rangle} = p_j \cup p_\sigma, j = 0, 1$.
    - WLOG we insist W_{2n} is always a split of R_{2n} so this case occurs infinitely.

Summarizing Construction

- $\bigcap R_n = \emptyset$ (infinitely often we lose the least element).
- $p_f = \bigcup_{\sigma \subset f} p_\sigma$ is a permutation of ω
- Images of c.e. sets are given by finitely many computable permutations on disjoint computable sets.
- R_{k+1} isn't split by $\{W_i | i \leq k\}$ so we can redefine/extend permutation on R_{k+1}.

Remark

Nifty but as Soare points out if $p[A] = B$ built in this fashion then $(p_1 \circ p_2 \circ \ldots \circ p_n)[A] = B$.

Permutations and Automorphisms

Question

Are all automorphisms of $\mathscr{E}^*$ induced by a permutation?

Remark

Since permutations respect $\subseteq$ this would show every $\boldsymbol{\Upsilon}^* \in \operatorname{Aut} \mathscr{E}^*$ is induced by some $\boldsymbol{\Upsilon} \in \operatorname{Aut} \mathscr{E}$.

Theorem (?)

Every automorphism $\boldsymbol{\Upsilon}(W_e) = W_{\upsilon(e)}$ is induced by a permutation $p \leq_{\mathbf{T}} \upsilon(e) \oplus 0'$.

Proof Idea

Idea

After some point map x to y only if for all $i \leq n$
$x \in W_i \iff y \in W_{\upsilon(e)}$.

Definition

The e-state (e-hat-state) of x is $\sigma(e, x)$ ($\hat{\sigma}(e, x)$) where:

$$\sigma(e, x) = \{i \leq e | x \in W_i\}$$
$$\hat{\sigma}(e, x) = \left\{i \leq e \middle| x \in W_{\upsilon(i)}\right\}$$

Proof

- At stage $2n$ define $p(x)$ for least $x \notin \operatorname{dom} p$.
- Let e_{2n} be max s.t. $(\exists y)(y \notin \operatorname{rng} p \wedge \sigma(e,x) = \hat{\sigma}(e,y))$.
- Let $p(x)$ be least such y.
- At odd stages define $p^{-1}(y)$ for least $y \notin \operatorname{dom} p^{-1}$.
- $\liminf_{n\to\infty} e_n = \infty$ so $p(W_e) =^* W_{v(e)}$
  - $|W_i| < \omega$ then eventually $W_i \subseteq \operatorname{dom} p, \operatorname{rng} p$

Advanced Automorphism Techniques

- Often we have A, B and want to build Υ with $\Upsilon(A) = B$.
- Difficult to directly build permutation with $p[A] = B$ while sending c.e. set to c.e. sets.
- Easier to work in $\mathscr{E}^*$ and effectively construct $W_{\upsilon(e)}$.
- Problem is respecting $\subseteq^*$.
  - Must ensure that $W_{\upsilon(e)}$ has same lattice of c.e. subsets/supersets as W_e.
  - Have to build $W_{\upsilon(e)}$ without knowing if $|W_e \cap A| = \infty, W_e \supseteq A, W_e \subseteq A$ or $W_e \supseteq \overline{A}$ at any stage.
  - To ensure $\Upsilon(W_e)$ is c.e. we need a somewhat effective grip on Υ

The Extension Theorem and Δ^0_3 Automorphisms

Definition

$\mathscr{L}(A)$ are the c.e. sets containing A and $\mathscr{E}(A)$ are the c.e. sets contained in A (under inclusion).

- Want to build automorphism Υ with $\Upsilon(A) = B$.
- The Extension Theorem (Soare) and Modified Extension Theorem (Cholak) break up construction.
  - Build (sufficiently effective) automorphism of $\mathscr{L}^*(A)$ with $\mathscr{L}^*(B)$.
  - Ensure (roughly) that (mod finite) elements fall into A and B in same e-state, e-hat-state.
- The Δ^0_3 automorphism method uses a complicated Π^0_2 tree construction to build Δ^0_3 automorphisms.

Some Results

- (Martin) h.h.s. sets and complete sets aren't invariant.
- (Soare) The maximal sets form an orbit
- (Downey, Stob) The hemi-maximal sets form an orbit.
- (Cholak, Downey, and Herrmann) The Hermann sets form an orbit.
- (Soare) Every (non-computable) c.e. set is automorphic to a high set.
- Hodgepodge of results about orbits of other classes of sets.

Completeness

Question

Is every W_e automorphic to a Turing complete r.e. set?

Theorem (Harrington-Soare)

There is an $\mathscr{E}$ definable property $Q(A)$ satisfied only by incomplete sets.

Theorem (Cholak-Lange-Gerdes)

There are disjoint properties $Q_n(A), n \geq 2$ satisfied only by incomplete sets.

$\mathscr{D}$-maximal sets

Definition (Sets disjoint from A)

$$\mathscr{D}(A) = \{B : \exists W (B \subseteq^* A \cup W \text{ and } W \cap A =^* \emptyset)\}$$

Let $\mathscr{E}_{\mathscr{D}(A)}$ be $\mathscr{E}$ modulo $\mathscr{D}(A)$, i.e., $B = C \mod \mathscr{D}(A)$ if

$$(\exists D_1, D_2 \text{ s.t. } D_1 \cap A =^* D_2 \cap A =^* \emptyset)[B \cup A \cup D_1 =^* C \cup A \cup D_2]$$

Definition

1. A is *hh-simple* iff $\mathscr{L}^*(A) = \{B \mid B \supset^* A\}$ is a (Σ^0_3) Boolean algebra.
2. A is $\mathscr{D}$*-hhsimple* iff $\mathscr{E}_{\mathcal{D}(A)}$ is a (Σ^0_3) Boolean algebra.
3. A is $\mathscr{D}$*-maximal* iff $\mathscr{E}_{\mathscr{D}(A)}$ is the trivial Boolean algebra iff

   $$(\forall B)(\exists D \text{ s.t. } D \cap A =^* \emptyset)[B \subset^* A \cup D \text{ or } B \cup A \cup D =^* \omega].$$

Definition

A is *$\mathscr{D}$-maximal* if

$$(\forall B)(\exists D \text{ s.t. } D \cap A =^* \emptyset)[B \subset^* A \cup D \text{ or } B \cup A \cup D =^* \omega].$$

Example

Maximal and hemi-maximal sets are $\mathscr{D}$-maximal.

A set that is maximal on a computable set is $\mathscr{D}$-maximal.

Question

- What are the orbits of $\mathscr{D}$-maximal sets?
- Do they form finitely many orbits?

$\mathscr{D}$-maximal sets and r-maximal sets

- Many ways to get $\mathscr{D}$-maximal sets already covered.
- (Proper) splits of maximal sets are in a single orbit.
- Maximal subsets of computable sets in a single orbit.
- (Cholak-Harrington) $\mathscr{D}$-maximal sets in A-special lists fall in a single orbit.
- Full consideration leaves only the case of $\mathscr{D}$-maximal sets contained in atomless r-maximal sets as potential source for infinitely many orbits.

Infinitely many orbits for $\mathscr{D}$-maximal sets

- Borrow technique from Nies and Cholak for showing atomless r-maximal sets aren't automorphic.
- Technique reveals structure of $\mathscr{L}^*(A)$ by giving it tree structure.
- In particular $\mathscr{L}^*(A)$ has structure T if there is a homomorphism ϕ from $\langle \mathscr{L}^*(A) - \omega, \subset_\infty \rangle$ to $\langle T, \subset \rangle$ s.t.

$$|W_e \bigcap \phi^{-1}(\sigma) - \phi^{-1}(\sigma^-)| = \infty \implies |W_e \bigcap \phi^{-1}(\sigma^-) - \phi^{-1}(\sigma^{--})| = \infty$$

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- Define infinite sequence of trees T^n which guarantee that incompatible structure of supersets.

Non-automorphic $\mathscr{D}$-maximal sets

- Build A $\mathscr{D}$-maximal subset of r-maximal set A_r with structure T^n.
- Build B $\mathscr{D}$-maximal subset of r-maximal set B_r with structure T^{n+1}.
- If $\boldsymbol{\Upsilon}(A) = B$ then there is superset of B_r with structure given by subtree of T^n.
- Incompatibility result ensures this is impossible.
- Gives us infinite number of orbits for $\mathscr{D}$-maximal sets.