On VC-Minimality in Algebraic Structures

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Outline

Directed Sets

Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$.

Definition.

We say that $\mathcal{C}$ is *directed* if, for all $B_0, B_1 \in \mathcal{C}$, we have that

1. $B_0 \subseteq B_1$,
2. $B_1 \subseteq B_0$, or
3. $B_0 \cap B_1 = \emptyset$.

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Definition.

Let $\Psi = \{\psi_i(x; \overline{y}_i) : i \in I\}$ be a set of partitioned formulas in a theory T. We say that Ψ is *directed* if the instances of Ψ are directed.

VC-Minimal theories

Definition (Adler).

A theory T is *VC-minimal* if there exists a family of formulas

$$\Psi(x) = \{\psi_i(x; \overline{y}_i) : i \in I\}$$

that is directed and every one-variable formula is T-equivalent to a boolean combination of instances of Ψ. We call Ψ the *generating family*.

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Strongly minimal, o-minimal, and weakly o-minimal theories are VC-minimal. Additionally, the theory of algebraically closed valued fields in the language $L = \{+, \cdot, 0, 1, |\}$ is VC-minimal. Every VC-minimal theory is dp-minimal.

Problems with VC-Minimality

It is hard to show directly that a theory is not VC-minimal.

Open Question.

Is VC-minimality closed under reducts (allowing parameters in the generating family)?

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Is VC-minimality closed under reducts (allowing parameters in the generating family)?

Are there natural examples of theories that are dp-minimal but not VC-minimal?

Convexly Orderable

Definition.

We say that a structure M is *convexly orderable* if there exists $\trianglelefteq$ a linear ordering on M such that, for all L-formulas $\varphi(x; \overline{y})$, there exists K_φ such that, for all $\overline{b} \in M^{\lg(\overline{y})}$, the set $\varphi(M; \overline{b})$ is a union of at most K_φ $\trianglelefteq$-convex subsets of M.

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Examples.

The following structures are convexly orderable:

1. Any o-minimal structure (e.g., $(\mathbb{Q};<)$),
2. Any structure with a weakly o-minimal theory,
3. Any strongly minimal structure (e.g., $(\mathbb{C},+,\cdot)$).

Lemmas on Convex Orderability

Lemma.

1. If M is convexly orderable and $N \equiv M$, then N is convexly orderable.
2. If M is a convexly orderable L-structure and $L_0 \subseteq L$, then $M|_{L_0}$ is convexly orderable.
3. If M is convexly orderable, $X \subseteq M$ is definable, and $E \subseteq M^2$ is a definable equivalence relation on X, then X/E with the induced structure is convexly orderable.

Connections

Lemma.

If X is a set and $\mathcal{C} \subseteq \mathcal{P}(X)$ is directed, then there exists a linear ordering $\trianglelefteq$ on X such that every $B \in \mathcal{C}$ is a $\trianglelefteq$-convex subset of X.

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Theorem (G., Laskowski).

If T is VC-minimal and $M \models T$, then M is convexly orderable.

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Open Question.

Is convexly orderable equivalent to VC-minimal?

Ordered Structures

Lemma.

If $M = (M; <, ...)$ is a linearly ordered structure that is convexly orderable, then there do not exist $X_0, X_1, ... \subseteq M$ pairwise disjoint definable that are each $<$-coterminal.

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If $M = (M; <, \ldots)$ is a linearly ordered structure that is convexly orderable, then there do not exist $X_0, X_1, \ldots \subseteq M$ pairwise disjoint definable that are each $<$-coterminal.

Theorem (Flenner, G.).

Let $(G; \cdot, <)$ be an ordered group and let $T = \mathrm{Th}(G; \cdot, <)$. The following are equivalent:

1. T is o-minimal,
2. T is VC-minimal,
3. G is convexly orderable,
4. G is abelian divisible.

Presburger Arithmetic

Corollary.

The structure $(\mathbb{Z}; +, <)$ is not convexly orderable. Hence, the theory of Presburger arithmetic is dp-minimal but not VC-minimal.

This was first discovered by Andrews, Cotter, and Freitag.

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Corollary.

The *p*-adic valued field $(\mathbb{Q}_p; +, \cdot, |)$ is not convexly-orderable. Hence, the theory of the *p*-adics is dp-minimal but not VC-minimal.

Ordered Fields

Corollary.

If $(F; +, \cdot, <)$ is an ordered field that is convexly orderable, then every positive element has an nth root for all n.

Proof.

Look at $(F_+; \cdot, <)$ and apply the previous corollary to show it is divisible. □

Open Question.

Are all convexly orderable ordered fields real closed?

An Example

Example.

Not all convexly orderable ordered structures are weakly o-minimal. For example, take $\mathbb{Q}$ and consider $D \subseteq \mathbb{Q}$ some dense, co-dense subset. Look at the structure $(\mathbb{Q}; <, D)$. Clearly this is not weakly o-minimal. However, it is convexly orderable and even VC-minimal. For Ψ, take

$$\Psi = \{D(x), D(x) \wedge x < y, \neg D(x) \wedge x < y, x = y\}.$$

Abelian Groups

For this section, let $(A;+)$ be an abelian group and let $T = \mathrm{Th}(A;+)$. Look at $\mathrm{PP}(A)$ the lattice of p.p.-definable subgroups of A. Consider the quasi-order on $\mathrm{PP}(A)$ given by

$$B_0 \precsim B_1 \text{ iff. } [B_0 : B_0 \cap B_1] < \aleph_0.$$

Let $\widetilde{\mathrm{PP}}(A) = \mathrm{PP}(A)/\sim$, where $\sim$ is the equivalence class generated by $\precsim$.

Theorem (Aschenbrenner, Dolich, Haskell, MacPherson, Starchenko).

T is dp-minimal if and only if $(\widetilde{\mathrm{PP}}(A);\precsim)$ is a linear order.

VC-minimality for Abelian Groups

Definition.

For $X \in \widetilde{\mathrm{PP}}(A)$, we say that X has *upward coherence* if there exists $B \in X$ such that, for all $C \in \mathrm{PP}(A)$ with $B \precsim\!\!\!\!/\; C$, $B \subseteq C$.

Theorem (Flenner, G.)

The following are equivalent:

1. T is VC-minimal,
2. A is convexly orderable,
3. T is dp-minimal and, for all $X \in \widetilde{\mathrm{PP}}(A)$, X has upward coherence.

Examples of Abelian Groups

Examples.

The theory of the following groups in the pure group language are VC-minimal:

1. $\mathbb{Z}$.
2. $\mathbb{Z}(p^\infty)$ for prime p.
3. $\mathbb{Z}_{(p)}$ for prime p.
4. $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$ for $k > 0$ and prime p.

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Examples.

The theory of the following group is dp-minimal but not VC-minimal:

$$A = \bigoplus_{n>0} \left(\mathbb{Z}/p^n\mathbb{Z}\right).$$

Quasi VC-minimality

Definition.

We say a theory T is *quasi-VC-minimal* if there exists Ψ a directed family such that all one-variable formulas are T-equivalent to a boolean combination of instances of Ψ and 0-definable formulas.

Example.

The theory of Presburger arithmetic, $T = \mathrm{Th}(\mathbb{Z}; +, <)$, is quasi VC-minimal.

Valued Fields

Lemma

If T is quasi-VC-minimal, $M \models T$, and $\varphi(x; \bar{y})$ is any formula, then there exists $\trianglelefteq_\varphi$ a linear ordering on M and $K_\varphi < \omega$ such that all instances of φ are a union of at most K_φ $\trianglelefteq_\varphi$-convex subsets of M.

So quasi-VC-minimal theories have "local convex orderability."

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So quasi-VC-minimal theories have "local convex orderability."

Theorem (Flenner, G.).

Suppose that K is a field with valuation $v : K^\times \to \Gamma$. Suppose further that the theory $T = \mathrm{Th}(K; +, \cdot, |)$ is quasi-VC-minimal. Then, Γ is divisible.

Here we interpret $x|y = v(x) \leq v(y)$.

The p-adics

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The theory of the p-adics, $T = \mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$, is dp-minimal but not quasi-VC-minimal.

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The theory of the p-adics, $T = \mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$, is dp-minimal but not quasi-VC-minimal.

So the following are all strict implications

VC-Minimal ⇒ Quasi-VC-Minimal ⇒ dp-Minimal

The End

Thank you for your time!