Reverse Mathematics and Field Extensions

Preliminary Report

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Reverse field theory

In the reverse math setting (second order arithmetic with limits on comprehension and induction) a field is a countable set with operations that satisfy the usual field axioms. One can encode copies of familiar fields like $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{2})$.

If every non-constant polynomial in K has a root in K, we say K is algebraically closed. An algebraic closure of F is an algebraically closed field $\overline{F}$ with an embedding $\varphi : F \to \overline{F}$.

$\mathrm{RCA}_0 \vdash$ *every field has an algebraic closure.*
RCA_0: recursive comprehension axiom

$\mathrm{WKL}_0 \leftrightarrow$ *algebraic closures are unique.*
WKL_0: weak König's lemma

$\mathrm{ACA}_0 \leftrightarrow$ *fields are subsets of their algebraic closures.*
ACA_0: arithmetic comprehension axiom

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$\mathrm{ACA}_0 \leftrightarrow$ *fields are subsets of their algebraic closures.*

These results appear in Friedman, Simpson, and Smith's paper [1] and also in Simpson's book [5]. They are related to earlier results in recursive (computable) algebra.

Extending automorphisms

For this talk, we will concentrate on characteristic 0 fields.

Theorem 1 (RCA_0) The following are equivalent:

(1) WKL_0.

(2) Let F be a field with an algebraic closure $\overline{F}$. If $\alpha \in \overline{F}$ and $\varphi : F(\alpha) \to F(\alpha)$ is an automorphism of $F(\alpha)$ that fixes F, then φ extends to an F-automorphism of $\overline{F}$.

Ideas from the proof of $(1) \to (2)$:

Build a tree of initial segments of F-automorphisms of $\overline{F}$.

At each node map $x \in \overline{F}$ to some root of some polynomial it satisfies. (Bounded levels.)

Stop extending initial non-automorphisms.

Any infinite path codes an F-automorphism.

Theorem 1 (RCA$_0$) The following are equivalent:

(1) WKL$_0$.

(2) Let F be a field with an algebraic closure $\overline{F}$. If $\alpha \in \overline{F}$ and $\varphi : F(\alpha) \to F(\alpha)$ is an automorphism of $F(\alpha)$ that fixes F, then φ extends to an F-automorphism of $\overline{F}$.

Ideas from the proof of $(2) \to (1)$:

Separate the ranges of disjoint positive injections f and g.

Let $F = \mathbb{Q}[\sqrt{p_{f(i)}}, \sqrt{2p_{g(i)}}]$, note that $\sqrt{2} \notin F$.

Define $\varphi : F(\sqrt{2}) \to F(\sqrt{2})$ by $\varphi(a + b\sqrt{2}) = a - b\sqrt{2}$.

Use (2) to extend φ to $\overline{\mathbb{Q}}$.

Since φ fixes F, $\{j \mid \varphi(\sqrt{p_j}) = \sqrt{p_j}\}$ includes the range of f and avoids the range of g.

Nontrivial automorphisms

Theorem 2 (RCA_0) The following are equivalent:

1. WKL_0.
2. Let F be a field and let K be a proper algebraic extension of F. Suppose that every irreducible polynomial over F that has a root in K splits into linear factors in K. Then there is a non-trivial F-automorphism of K.

Theorem (Metakides and Nerode [4]) There is a recursively presented field F with a recursively presented algebraic extension K such that K has many F-automorphisms, but the only computable F-automorphism is the identity.

Nontrivial automorphisms

Theorem 2 (RCA_0) The following are equivalent:

1. WKL_0.
2. Let F be a field and let K be a proper algebraic extension of F. Suppose that every irreducible polynomial over F that has a root in K splits into linear factors in K. Then there is a non-trivial F-automorphism of K.

Ideas from the reversal:

Separate the ranges of disjoint positive injections f and g.

Let $K = \mathbb{Q}(\sqrt{p_i} \mid i \in \mathbb{N})$.

Let $F = \mathbb{Q}(\sqrt{p_i}\sqrt{p_{(i,g(j))}}, \sqrt{p_{(i,f(j))}} \mid i, j \in \mathbb{N})$.

Prove that $\sqrt{2} \notin F$.

If φ is a non-identity F-autom. of K, it moves some $\sqrt{p_i}$.

For that value of i, $\{j \mid \varphi(\sqrt{p_{(i,j)}}) = \sqrt{p_{(i,j)}}\}$ includes the range of f and avoids the range of g.

Notions of normality

Here are several versions of "K is a normal extension of F." The first three are from Lang [3].

NOR1: Every irred. polynomial over F that has a root in K splits completely over K.

NOR2: K is the splitting field of some sequence of polynomials over F.

NOR3: If $\varphi : K \to \overline{F}$ is an F-embedding, then φ is an F-automorphism of K.

NOR4: If $\varphi : \overline{F} \to \overline{F}$ is an F-automorphism, then φ is an F-automorphism on K.

Thm 3: RCA_0 proves NOR1 $\leftrightarrow$ NOR2 $\to$ NOR3 $\to$ NOR4.

Thm 4 (RCA_0) **The following are equivalent:**

1. WKL_0
2. NOR4 $\to$ NOR2
3. NOR4 $\to$ NOR3
4. NOR3 $\to$ NOR2

Isomorphic towers

Theorem 5 (RCA$_0$) The following are equivalent:

1. ACA$_0$.
2. Suppose $K = \langle k_i \rangle_{i \in \mathbb{N}}$ and $J = \langle j_i \rangle_{i \in \mathbb{N}}$ are algebraic extensions of F. If for all $n \in \mathbb{N}$, $F(k_1, \ldots, k_n) \preceq_F J$ and $F(j_1, \ldots, j_n) \preceq_F K$, then $K \cong_F J$.

Theorem 6 (RCA$_0$) The following are equivalent:

1. WKL$_0$.
2. Let $\langle F(\vec{\alpha}_i) \mid i \in \mathbb{N} \rangle$ and $\langle F(\vec{\beta}_i) \mid i \in \mathbb{N} \rangle$ be increasing sequences of finite NOR1-normal algebraic extensions of F. Let $K = \bigcup_{i \in \mathbb{N}} F(\vec{\alpha}_i)$ and let $J = \bigcup_{i \in \mathbb{N}} F(\vec{\beta}_i)$. If for all $i \in \mathbb{N}$, $F(\vec{\alpha}_i) \preceq_F J$ and $F(\vec{\beta}_i) \preceq_F K$, then $K \cong_F J$.

The reversal for Theorem 6 is a construction of Miller and Shlapentokh.

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