Homology groups in model theory

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Outline

We define the notion of a homology group in a model-theoretic context.

The groups measure the failure of generalized amalgamation of an appropriate dimension.

The group H_2 is shown to be a certain automorphism group.

Plan:

- Example of a structure with a non-trivial group H_2
- Generalized amalgamation
- Simplices
- Homology group calculations

Example

It is possible that $\overline{ab} := \mathrm{acl}(ab)$ contains elements that are definable from $\overline{ac} \cup \overline{bc}$, but not definable from $\overline{a}\overline{b}$.

Fix a finite group G. Take a structure with two sorts:

- I an infinite set,
- $P := I^2 \times |G|$, where $|G|$ is a set.

Add a projection $\pi : P \to I^2$.

Example

Let $a, b \in I$. Then

- $[a] := \pi^{-1}(a, a)$,
- $[a, b] := \pi^{-1}(a, b)$,
- the symbol δ_{ab} implies $\delta_{ab} \in [a, b]$.

Relation θ on P^3 holds if and only if the elements have the form $(\delta_{bc}, \delta_{ac}, \delta_{ab})$ and (abusing notation) $\delta_{ab} \cdot \delta_{bc} = \delta_{ac}$.

Example

Note that θ defines:

- a group operation on $[a]$,
- the action of $[a]$ on $[a, b]$, and
- a way to compose δ_{ab} and δ_{bc}.

Facts (Goodrick, Kim, K.)

1. *The theory of the above structure is totally categorical.*
2. *The group G is abelian if and only if for any $a \neq b \in I$ and for all $\gamma, \delta \in [a, b]$ we have $\mathrm{tp}(\gamma/[a][b]) = \mathrm{tp}(\delta/[a][b])$.*

Example

From this point, $G = (G, +)$ is an abelian group. The automorphism group $\mathrm{Aut}([a, b]/[a][b])$ is isomorphic to G.

The structure described above is a *definable connected finitary abelian groupoid with the vertex group* G. The set I is the set of objects, P is the set of morphisms, θ gives the composition.

Groupoid axioms are routine to check; associativity is interesting.

Associativity is equivalent to the following:

If $\delta_{cd} \circ \delta_{bc} = \delta_{bd}$, $\delta_{cd} \circ \delta_{ac} = \delta_{ad}$, and $\delta_{bd} \circ \delta_{ab} = \delta_{ad}$,

then $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$.

"θ on three sides implies θ on the fourth."

Generalized uniqueness and existence

2-uniqueness is stationarity: for independent a, b, the type of $\mathrm{acl}(ab)$ is determined by the types of $\mathrm{acl}(a)$, $\mathrm{acl}(b)$.

3-uniqueness is more subtle:

Choose distinct $a, b, c \in I$ and fix δ_{ab}, δ_{bc} and δ_{ac} such that $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$.

Take a non-identity automorphism σ of $[a, c]$. Then necessarily $\delta_{bc} \circ \delta_{ab} = \sigma(\delta_{ac})$ fails.

We get non-isomorphic ways of embedding the "sides" $[a, b]$, $[b, c]$ and $[a, c]$ into a "triangle":

1. use the identity embeddings (I will denote this object $[a, b, c]$);
2. twist one of the sides by an automorphism (I will denote this by $[\overline{a, b, c}]$).

Generalized uniqueness and existence

3-existence is the Independence Theorem.

4-existence:

In the example, we are not able to find a joint realization (are not able to amalgamate) four types that express the following:

1. $\delta_{cd} \circ \delta_{bc} = \delta_{bd}$,
2. $\delta_{cd} \circ \delta_{ac} = \delta_{ad}$,
3. $\delta_{bd} \circ \delta_{ab} = \delta_{ad}$,
4. $\delta_{bc} \circ \delta_{ab} \neq \delta_{ac}$.

As usual, δ_{xy} is an element in the fiber $[x, y]$.

Generalized uniqueness and existence require tracking the embeddings of lower-dimensional parts into the higher-dimensional ones.

Simplices

This is formalized by the notion of an *n-simplex*. Fix a type p.

Definition

Let $\mathcal{C}$ be the category of algebraically closed subsets of the form $\mathrm{acl}(a_0, \ldots, a_n)$ for some $n \geq 0$ and a_i, $i \leq n$, are independent realizations of p. Morphisms are elementary embeddings.
An n-simplex is a functor $f : \mathcal{P}(s) \to \mathcal{C}$, for $s \subset \omega$, $|s| = n + 1$, such that

1. for all non-empty $u \in \mathcal{P}(s)$, we have $f(u) = \mathrm{acl}(\bigcup_{i \in u} f_u^{\{i\}}(\{i\}))$ and
2. if $w \in \mathcal{P}(s)$ and $u, v \subseteq w$, then

$$f_w^u(u) \underset{f_w^{u\cap v}(u\cap v)}{\downarrow} f_w^v(v).$$

Caution: bases!

Simplices

In the example:

- 0-simplices: $[a]$, $a \in I$;
- 1-simplices: $[a, b]$, $a \neq b \in I$;
- 2-simplices: $[a, b, c]$, $[\overline{a, b, c}]$, $\ldots$

S_n is the collection of all n-simplices.

C_n is the free abelian group generated by S_n.

Homology groups

If $n \geq 1$, $f = [a_0, \ldots, a_n]$ is an n-simplex, and $0 \leq i \leq n$, then

- $\partial_n^i(f) = [a_0, \ldots, \widehat{a_i}, \ldots, a_n]$;
- $\partial_n(f) = \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(f)$.

In particular,

$$\partial[a, b, c, d] = [b, c, d] - [a, c, d] + [a, b, d] - [a, b, c].$$

Z_n is the set of all chains in C_n whose boundary is 0.

B_n is the set of all chains in C_n of the form $\partial(c)$ for some $c \in C_{n+1}$.

$H_n = Z_n / B_n$.

Examples of chains

In the example, the 1-chain of the form

$$[a, b] + [b, c] + [c, d] - [a, d]$$

is a 1-cycle. It is also a 1-boundary because it is the boundary of the chain

$$[a, b, e] + [b, c, e] + [c, d, e] - [a, d, e].$$

Note that each of the 2-simplices above can be constructed using 3-existence.

The 2-chain $[b, c, d] - [a, c, d] + [a, b, d] - [a, b, c]$ is a 2-boundary, but

$$[b, c, d] - [a, c, d] + [a, b, d] - [\overline{a, b, c}]$$

is a 2-cycle, but not a 2-boundary. We call such cycles *2-shells*.

Computing H_2

Theorem (Goodrick, Kim, K.)

If T has $\leq (n+1)$-existence for some $n \geq 1$, then

$$H_n = \{[c] \mid c \text{ is an } n\text{-shell with support } \{0, \ldots, n+1\}\}.$$

Steps:

1. Show that an n-cycle is a linear combination of n-shells, up to ∂;
2. Show how to move n-shells into a single one, up to a boundary.

So if T has also $(n+2)$-existence, then H_n is trivial. In particular, a stable T with 4-existence has trivial H_2.

Computing H_2

What about the converse? What types of groups can we have as H_2?

Theorem (Goodrick, Kim, K.)

(T stable.) We have $H_2(p) = \mathrm{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1})$ *where* $\{a_0, a_1, a_2\}$ *are independent realizations of p and*

$$\widetilde{a_0 a_1} := \overline{a_0 a_1} \cap \mathrm{dcl}(\overline{a_0 a_2}, \overline{a_1 a_2}).$$

Moreover $H_2(p)$ *is always an abelian profinite group. Conversely any abelian profinite group can occur as* $H_2(p)$*.*

Fact (Goodrick, K.)

If stable T fails 4-existence, then there is a type p and independent realizations a_i *of p such that* $\mathrm{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1})$ *is non-trivial.*

Computing H_2

How do we know that $[b,c,d]-[a,c,d]+[a,b,d]-[\overline{a,b,c}]$ is not a boundary?

Fix elements $\delta_{xy} \in [x,y]$ for $x,y \in \{a,b,c,d\}$. These elements are embedded into the 2-simplices; denote the images by δ_{xy}^{xyz}.

A 2-shell is a boundary if and only if for some (any) choice of δ's we have

$$\begin{aligned}(\delta_{cd}^{bcd}-\delta_{bd}^{bcd}+\delta_{bc}^{bcd})-(\delta_{cd}^{acd}-\delta_{ad}^{acd}+\delta_{ac}^{acd})\\ +(\delta_{bd}^{abd}-\delta_{ad}^{abd}+\delta_{ab}^{abd})-(\delta_{bc}^{abc}-\delta_{ac}^{abc}+\delta_{ab}^{abc})=0.\end{aligned}$$

Next steps

First-order:

Conjecture

If T is stable with $\leq (n+1)$-existence, then

$$H_n(p) = \operatorname{Aut}(\widetilde{a_0 \ldots a_{n-1}} / \bigcup_{i=0}^{n-1} \overline{\{a_0 \ldots \hat{a}_i \ldots a_{n-1}\}}).$$

Non-elementary:

In [Goodrick,Kim,K.], the definitions are stated for a general context: functors into a category satisfying certain properties. What happens if the category is the class of atomic models?