COMBINATORIAL PROPERTY VS COMPUTATIONAL PROPERTY

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ABSTRACT. A set C can be strongly coded under condition $< \mathcal{B}, \mathcal{C}(A) >$, where $\mathcal{B}$ and $\mathcal{C}$ are classes of sets possibly with other parameters, iff there exists $A \in \mathcal{B}$ such that every $Z \in \mathcal{C}(A)$ can be used to compute C. The issue is widely studied especially in effective mathematics and reverse mathematics. In this paper, we focus on three kinds of conditions, namely, density condition, enumeration condition and partition condition. For density condition and enumeration condition, we give necessary and sufficient conditions for the parameters that ensure, under the corresponding coding condition $< \mathcal{B}, \mathcal{C}(A) >$ any set can be strongly computed. As a corollary, we show that for any given $C >_T 0$, if we restrict A to have at least constant density on each member of a computable array of mutually disjoint finite sets then there exists an infinite subset of A that can not be used to compute C. This is in contrast with a well-known result that if A is allowed to have density that approaches to 0, then for any C there exists A such that C can be computed by any infinite $G \subseteq A$. In addition we give a simplified proof of a main theorem in Greenberg and Miller [5] using a combinatorial result used in the proof of above theorem. As to enumeration condition we also give necessary and sufficient condition for a degree that can be strongly coded under corresponding condition. The last condition we study is partition condition. We give applications of our results including RT^2_2 does not imply WWKL_0.

1. Introduction

An important issue in computer science is how to code and extract information in an robust way. These questions can also be expressed in computability theory, namely, how to code (compute) a set by an "object", i.e. any member presenting this object can compute C. The paradigm is in general as following,

Definition 1.1. Say we can strongly code a set C (a class of sets $\mathcal{C}$) under condition $< \mathcal{B}, \mathcal{C}(A) >$ iff there exists $A \in \mathcal{B}$ such that $\mathcal{C}(A) \geq_u$

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$\{C\}$ (or $\mathcal{C}(A) \geq_u \mathcal{C}$), where $\geq_u$ is Muchnik reducibility where $\mathcal{B}$ and $\mathcal{C}(A)$ are classes that may depend on other "parameters".

This issue is widely studied in branches such as effective mathematics, reverse mathematics etc. The following are some examples.

In [7],$\mathcal{C}$ = members of a Π^0_1 class of 2^ω, ($\mathcal{B} = 2^\omega$).

Theorem 1.2 (Jockusch and Soare)**.** *For any infinite computable tree $T \subseteq 2^{<\omega}$, and any given non-computable degree a, there exists $f \in [T]$ such that $a \nleq f$.*

In [11] $\mathcal{B}$ =presentations of a class of some model, $\mathcal{C}(A)$ = presentation that is isomorphic to A.

Theorem 1.3 (Richter)**.** *For any of the following kind of structures: Graph, Lattice, Abel group, we have that for any given Turing degree a, there exists a model of that kind, such that $\mathcal{C}(A) \geq_u a$.*

In [12] $\mathcal{B}$ = all computable 2-colorings, $\mathcal{C}(f)$ = infinite homogeneous sets of the 2-coloring f.

Theorem 1.4 (Seetapun)**.** *For any computable 2-coloring f and any non-computable degree a, there exists an infinite homogeneous set of f G, such that $a \nleq G$.*

In [10] $\mathcal{B}$ = presentations of a continuous function, $\mathcal{C}(A)$ = presentations of the function represented by A.

Theorem 1.5 (Miller)**.** *There exists $\lambda_0 \in \mathcal{B}$, such that for every $\lambda \in \mathcal{C}(\lambda_0)$ there exists $\gamma \in \mathcal{C}(\lambda_0)$ such that $\lambda \nleq \gamma$.*

In [3], $\mathcal{B} = 2^\omega, \mathcal{C}(A) = \{X \in 2^\omega : X \subseteq A \vee X \subseteq \overline{A}, |X| = \infty\}$

Theorem 1.6 (Dzhafarov and Jockusch)**.** *For any non-computable degree a, and any set A, there exists $G \in \mathcal{C}(A)$ such that $a \nleq G$.*

Sometimes, we study whether one could even cone avoid (or code) a class of degrees rather than a given degree. The following are some examples.

Theorem 1.7 (Miller and Greenberg [5])**.** *If $j : \omega \to \omega \setminus \{0, 1\}$ is a recursive nondecreasing and unbounded function, then there is a $f \in DNR_j$ that does not compute any 1-random.*

In the following two sections we introduce without detailed proof of some results from the author's [9] and [8]. Section two studies density conditions and we obtain a result in contrast with a classical well known coding result. We also demonstrate that using a combinatorial lemma used in the proof of a coding result, we are able to give another proof (possibly easier) of a core theorem of [5] (theorem 1.7). This resembles the theorem of Downey, Greenberg, Jockusch and Milans [2]. Section three studies enumeration conditions, we give another characterization of hyperarithmatic degree other than the beautiful one given by Solovay [13]. Section four studies partition conditions, the result has important applications in reverse mathematics and algorithmic complexity theory. What is notable is that in that lemma the condition is purely combinatorial.

2. Coding under density conditions

In this section we study the following coding condition.

Definition 2.1.
(1) Let $\{S_n\}_{n\in N}$ be a strong array of mutually disjoint finite sets, $\lim_{n\to\infty} |S_n| = \infty$
(2) Call a function $\varepsilon : \omega \to \mathbb{R}$ density function iff $(\forall n)\varepsilon(n) \in (0, 1)$. Denote by $\varepsilon, \delta \cdots$ functions and $\varepsilon_0, \delta_0 \cdots$ constants.
(3) For two functions ε, δ, write $\varepsilon \preceq \delta$ iff $(\forall n)[\varepsilon(n) \leq \delta(n)]$.

Definition 2.2.

$$\mathcal{B}_1(\varepsilon) = \{A \in 2^\omega : (\forall n)\ \frac{|A \cap S_n|}{|S_n|} > \varepsilon(n)\}$$

$$\mathcal{C}_1(A, \varepsilon, \delta) = \{Z \in 2^\omega : Z \subseteq A \text{ is infinite, and } (\forall n)\ Z \cap S_n \neq \emptyset \Rightarrow \frac{|Z \cap S_n|}{|S_n|} > \delta(n)\}$$

A classical result said, in terms of the above definition,

Proposition 2.3 (Dekker and Myhill)**.** *For any computable density function ε, if $\lim_{n\to\infty} \varepsilon(n) = 0$ then we can strongly code any C under condition $< \mathcal{B}_1(\varepsilon), \mathcal{C}_1(A, \varepsilon, 0) >$, i.e. for any set C there exists a set A, $(\forall n)\ \frac{|A \cap S_n|}{|S_n|} \geq \varepsilon$, s.t. for any infinite set $G \subseteq A$, we have $G \geq_T C$.*

In contrast, the following result shows that condition $\lim_{n\to\infty} \varepsilon(n) = 0$ can not be removed if δ is not bounded away from 0. But if $(\forall n)\delta(n) > \delta_0 > 0$ for some constant δ_0, then we can still strongly code C.

Theorem 2.4. *For any $C >_T 0$, and a constant density function ε_0, we can strongly code C under condition $< \mathcal{B}_1(\varepsilon_0), \mathcal{C}_1(A, \varepsilon_0, \delta) >$, where δ is a computable density function satisfying $(\forall n)\varepsilon_0 > \delta(n) > 0$, if and only if $\delta(n)$ is bounded away from 0, i.e. there exists $\delta_0 > 0$, $(\forall n)\delta(n) > \delta_0$.*

Actually, we can obtain a necessary and sufficient condition for computable density function ε, δ to ensure that any given set C can be strongly coded under the corresponding condition.

Theorem 2.5. *We can strongly code any given non-computable degree C under condition $< \mathcal{B}_1(\varepsilon), \mathcal{C}_1(A, \varepsilon, \delta) >$, where ε, δ are computable density functions, if and only if the following hold:*

(1) $(\forall \varepsilon' > 0)(\exists \delta' > 0)$ *such that* $\varepsilon(n) > \varepsilon' \Rightarrow \delta(n) > \delta'$;
(2) $(\exists \gamma > 0)\ \dfrac{1-\varepsilon(n)}{1-\delta(n)} > \gamma$.

Sketch proof. The if direction uses Mathias forcing.

To prove the only if direction, i.e. the coding method, the following combinatorial result is the core.

Definition 2.6. For a finite set W, a $\varepsilon - \delta - k-$disperse class $\{B_i\}_{i \leq m}$ is a finite class of finite subsets of W, such that each is of size at least $[\varepsilon|W|] + 1$, the intersection of any k members of $\{B_i\}$ has size at most $[\delta|W|] + 1$. $\{B_i\}_{i \leq m}$ is a *maximal* $\varepsilon - \delta - k-$disperse class of W, iff for any $\varepsilon - \delta - k-$disperse class $\{B'_j\}_{j \leq n}$ (of W), $m \geq n$ holds. Let $m(\varepsilon, \delta, k, N) = max\{n < 2^N :$ there exists an $n-$ size $\varepsilon - \delta - k -$ disperse class of $\{1, 2 \ldots N\}\}$

Lemma 2.7. *For any $0 < \delta < \varepsilon < 1$, there exists an integer k, such that $\lim_{N \to \infty} m(\varepsilon, \delta, k, N) = \infty$. Moreover k can be effectively computed from ε, δ. (Denote by $k(\varepsilon, \delta)$ the minimal integer k that ensures $m(\varepsilon, \delta, k, N)$ to approach to infinite.)*

The idea of the coding is similar to error correcting code. For example suppose, $\varepsilon_0 > \varepsilon(n) > \delta(n) > \delta_0 > 0$ for some constant ε_0, δ_0. Let $k = k(\varepsilon_0, \delta_0)$. We construct on each S_m an n-size $\varepsilon_0 - \delta_0 - k$-disperse class where $n = m(\varepsilon_0, \delta_0, k, |S_m|)$, intuitively each member of the disperse class corresponds to a coded information, say a string of w-length and $A \cap S_m$ is the right one i.e. $C \upharpoonright w$ (C is the given set). If $Z = \{n : G \cap S_n \neq \emptyset\}$ then since $G \cap S_m$ has density larger than δ_0 so at most k member of the disperse classes contains $G \cap S_m$ i.e. we could compute a k-enumeration of C, which can be used to compute C.

Proof of Lemma 2.7. First it is shown that if for all $1 > \varepsilon > \delta > 0$, there exists k such that for all $n \in N$ the following group of linear inequalities have solutions, then the result follows. Let $\{x_\rho\}_{|\rho| \leq n}, \rho \neq 00 \ldots 0$ be $2^n - 1$ reals. Consider set of inequalities:

$$\forall \rho, \qquad x_\rho \geq 0 \tag{2.1}$$

$$\sum_\rho x_\rho \leq 1 \tag{2.2}$$

$$\forall 1 \leq i \leq n, \qquad \sum_{i \in set(\rho)} x_\rho > \varepsilon \tag{2.3}$$

$$\forall \rho, (|set(\rho)| \geq k) \Rightarrow \sum_{set(\sigma) \supseteq set(\rho)} x_\sigma < \delta \tag{2.4}$$

Let $1 > \varepsilon' > \varepsilon$, $0 < \delta' < \delta$, and let x_ρ be a solution of the above inequalities with ε, δ replaced by ε', δ'. Then, for any $\gamma > 0$ there exists an integer m such that $\forall N > m$, the $2^n - 1$ rational number $x'_\rho = \frac{[N x_\rho]}{N}$ satisfy the above inequalities with ε', δ' replaced by $\varepsilon' - \gamma, \delta' + \gamma$. Let γ be sufficiently small such that $\varepsilon' - \gamma > \varepsilon, \delta' + \gamma < \delta$. Given N sufficiently large, let $\{X_\rho\}_{i \leq 2^n - 1}$ be a class of $2^n - 1$ disjoint subsets of $\{1, 2, \ldots N\}$, such that $|X_\rho| = [x_\rho N]$ (by second inequality this class exists). The n size $\varepsilon - \delta - k-$disperse class of $\{1, 2 \ldots N\}$ $\{B_i\}_{i \leq n}$is as following, $B_i = \bigcup_{i \in set(\rho)} X_\rho$. It's easy to check $\{B_i\}$ is a $\varepsilon - \delta - k-$disperse class: $|B_i| > \varepsilon N$, besides for any k members of $\{B_i\}$, $\{B_r\}_{r \in K}$, it follows:

$$|\bigcap_{r \in K} B_r| = |\bigcup_{K \subseteq set(\rho)} X_\rho| = \sum_{K \subseteq set(\rho)} x'_\rho < \delta' + \gamma < \delta$$

Now we give a simple solution for the above inequalities with

$$k = [\frac{\log \delta}{\log \varepsilon}] + 1$$

Let

$$x_\rho = \begin{cases} \varepsilon^k (1 - \varepsilon)^k & \text{if } |set(\rho)| = k > 1 \\ \varepsilon (1 - \varepsilon)^{n-1} + \frac{1}{n}(1 - \varepsilon)^n & \text{if } |set(\rho)| = 1 \end{cases}$$

Clearly the first inequality is satisfied. Further more,

$$\begin{aligned} \sum_\rho x_\rho &= \sum_{|set(\rho)| = 1} x_\rho + \sum_{set(\rho) > 1} x_\rho \\ &= (1 - \varepsilon)^n + n\varepsilon(1 - \varepsilon)^{n-1} + \sum_{j=2}^n C_n^j \varepsilon^j (1 - \varepsilon)^{n-j} \qquad (2.5) \\ &= \sum_{j=0}^n C_n^j \varepsilon^j (1 - \varepsilon)^{n-j} = 1 \end{aligned}$$

i.e. the second inequality is satisfied. For $i \leq n$, let $set(\rho_i) = \{i\}$, then

$$
\begin{aligned}
\sum_{i \in set\rho} x_\rho &= x_{\rho_i} + \sum_{\substack{i \in set(\rho) \\ |set(\rho) > 1|}} x_\rho \\
&= \frac{1}{n}(1-\varepsilon)^n + \varepsilon(1-\varepsilon)^{n-1} + \sum_{j=1}^{n-1} C_{n-1}^j \varepsilon^{j+1}(1-\varepsilon)^{n-1-j} \qquad (2.6) \\
&> \varepsilon \sum_{j=0}^{n-1} C_{n-1}^j \varepsilon^j (1-\varepsilon)^{n-1-j} = \varepsilon
\end{aligned}
$$

i.e. the third inequality is satisfied. For $K \subset \{1, 2 \dots N\}$ that $|K| = k$,

$$\sum_{set(\rho) \supseteq K} x_\rho = \sum_{j=0}^{n-k} C_{n-k}^j \varepsilon^{k+j}(1-\varepsilon)^{n-k-j} = \varepsilon^k < \delta$$

Thus all inequalities are satisfied. □

□

What is interesting is that Lemma 2.7 can be used to give an another proof of a core theorem of Greenberg and Miller's paper [5], we assume the reader is familiar with that paper.

Theorem 2.8 (Greenberg and Miller)**.** *If $j : \omega \to \omega \setminus \{0, 1\}$ is a recursive nondecreasing and unbounded function, then there is a $f \in DNR_j$ that does not compute any 1-random.*

Sketch proof. We only give the core of the proof. We begin by generalize concept of n-bushy.

Definition 2.9. Let $p : \omega \to \omega$ be a positive partial function. A finite tree $T \subseteq j^{<\omega}$ is p-bushy above σ iff every element of T is comparable with σ and for every $\tau \in T$ that extends σ and is not a leaf of T there are at least $p(|\sigma|)$ immediate extension of τ in T. (Here we assume $p(n) = 0$ if p is not defined on n.)

The definition of n-big etc are naturally generalized to $p-$big according to above definition.

The combinatorial property of the bushy trees is,

Lemma 2.10. *Let $p, q : \omega \to \omega$ be two functions with domain $\{0, 1, \dots, n\}$. Let $B, C \subseteq j^{<\omega}$, if $B \cup C$ is $p + q - 1$-big, then either B is p-big or C is q-big.*

Lemma 2.11. *Let $p, q : \omega \to \omega$ be two functions with $dom(p) \supseteq dom(q)$, and $q \geq p$ on $dom(q)$. If C is not p-big over σ and B is $q-$big over σ then there exists $\tau \in B$ such that C is not p-big over τ.*

The forcing condition we use is (σ, B, ε) where the first two components are the same as [5], ε is a computable density function i.e. $(\forall n)1 > \varepsilon(n) > 0$, moreover, $(\forall n > |\sigma|)\varepsilon(n) << 1$; and B is not εj-big (εj is short for $\lambda n.\varepsilon(n)j(n)$) over σ. Furthermore, we require $\lim_{n\to\infty} \varepsilon(n) = 0$ and ε is total recursive.

Suppose we are given condition (σ, B, ε). The extending forcing condition will be (τ, C, ε') such that $\varepsilon' > 4\varepsilon$ and $\lim_{n\to\infty} \varepsilon(n)/\varepsilon'(n) = 0$. Let $Conv_\Phi(N) = \{\tau : (\forall x < N)\Phi^\tau(x) \downarrow\}$. If $Conv_\Phi(N)$ over σ is not $\varepsilon' j-$big then we are done by letting $(\tau, Conv_\Phi^\tau(N) \cup B^\tau, (\varepsilon' + \varepsilon))$ where $\tau \succeq \sigma$ is sufficiently large that ensure for $n \geq |\tau|$ $\varepsilon(n) + \varepsilon'(n) << 1$, B^τ denote $\{\rho \in B : \rho \succeq \tau\}$. Therefore we assume $(\forall N)Conv_\Phi(N)$ over σ is $\varepsilon' j$-big. Let

$$\mathcal{D}_\Phi = \{(\sigma, B, n) : (\sigma, B, n) \text{ is a condition such that } B \text{ is of } \varepsilon' j - big \text{ and } Conv_\Phi(N) \subseteq B\}$$

Now we can prove the core lemma.

Lemma 2.12. *Assume $(\forall N)Conv_\Phi(N)$ over σ is $\varepsilon' j$-big. Let $\delta > 0$ be any give small constant, let ε' be a recursive total density function, satisfying $\lim_{n\to\infty} \varepsilon(n)/\varepsilon'(n) = 0$ and $\varepsilon' > 4\varepsilon$. Then there exists $N \in \omega$ sufficiently large, and a $C \subseteq Conv_\Phi(N)$ that is $\varepsilon'' j$-big over σ where $\varepsilon'' + \varepsilon < \varepsilon'$, such that*

$$|\{\Phi^\tau \upharpoonright N : \tau \in C\}| \leq \delta \cdot 2^N$$

Sketch proof.

1. Choose $k = k(1 - \delta, \delta)$.
2. Choose an m such that $(\forall u \geq m)$ $\varepsilon'(u) - 2k\varepsilon(u) >> 2\varepsilon(u)$.
3. Choose an n such that every n number of $[\varepsilon' - 2\varepsilon]j$-big tree over σ up to level m there exists k number of them that are the same.
4. Choose an N such that $2^N > n$ and choose an $n-$size $(1 - \delta) - \delta - k$-disperse class of N-length strings, namely $\{B_i\}_{i \leq n}$.
5. Choose an M, such that there exists a $\varepsilon' j$-big tree over σ up to level M, namely C, such that $C \subseteq Conv_\Phi(N)$. (Such M exists because (σ, B, ϵ) has no extension in $\mathcal{D}_\Phi$.)

Let $Conv_\Phi(B) = \{\tau \in C : \Phi^\tau \in B\}$.

If for some B_i $Conv_\Phi(B_i)$ is not $(\varepsilon' - 2\varepsilon)j$-big, then $Conv_\Phi(\overline{B_i})$ is $2\varepsilon j$-big, therefore we can find $\rho \in Conv_\Phi(\overline{B_i})$ such that B (the open set in the given condition (σ, B, ε) is not εj-big over ρ, therefore let $C = Conv_\Phi^\rho(\bar{B}_i)$ and we are done since $Conv_\Phi^\rho(\bar{B}_i)$ is εj-big.

If all B_i, $Conv_\Phi(B_i)$ is $(\varepsilon' - 2\varepsilon)j$-big, select k of them, namely $B_{r_1}, B_{r_2}, \ldots, B_{r_k}$ such that $Conv_\Phi(B_{r_i})$ are all the same up to level m, let C be $\bigcap_{i=1}^{k} Conv_\Phi(B_{r_i})$, clearly C is of density $\varepsilon' - 2\varepsilon > 2\varepsilon$ up to level m and is of density larger than $\varepsilon' - 2k\varepsilon > 2\varepsilon$ from m to M. But on C $|\{\Phi^\tau \upharpoonright N : \tau \in C\}| < \delta \cdot 2^N$ since $\{B_i\}_{i \le n}$ is disperse class. So we could give every string in $\{\Phi^\tau \upharpoonright N : \tau \in C\}$ sufficiently small description.

□

□

In [2], also using combinatorial result resembles that of Lemma 2.7, it is shown that

Theorem 2.13. *1*-RAND $\not\leq_M$ DNR$_3$

Where $\leq_M$ denote Medevedev reducibility.

We now turn to the second version of density conditions.

Definition 2.14.

$$\mathcal{B}_1(\varepsilon) = \{A \in 2^\omega : (\forall n) \ \frac{|A \cap S_n|}{|S_n|} > \varepsilon(n)\}$$

$$\mathcal{C}_1'(A, \varepsilon, \delta) = \{Z \in 2^\omega : Z \subseteq A \text{ is infinite, and } (\forall n) \ \frac{|Z \cap S_n|}{|S_n|} > \delta(n)\}$$

Theorem 2.15. *For any set $C >_T 0$, we can strongly code C under condition $< \mathcal{B}_1(\varepsilon), \mathcal{C}'_1(A, \varepsilon, \delta) >$ if and only if $\exists \gamma > 0$ $\frac{1-\varepsilon(n)}{1-\delta(n)} > \gamma$ infinitely often.*

3. Coding under enumeration condition

Definition 3.1. $\mathcal{B}_2 = \{f \in \omega^\omega : \forall n, f(n) \in S_n\}$. $\mathcal{C}_2(f, k) = \{h \in \omega^\omega :$ h is a $k(n) -$ *enumeration* of $f\}$.

Another version is to consider all functions, i.e. $\mathcal{C}$ is the same but $\mathcal{B}_2' = \omega^\omega$.

Theorem 3.2. *We can strongly code any given $C >_T 0$ under condition $< \mathcal{B}_2, \mathcal{C}_2(f, k) >$ if and only if $(\exists m) k(n) < m$ infinitely often.*

We could give a characterization of hyperarithmetic degree in terms of coding under condition,

Theorem 3.3. *A given set C can be strongly coded under condition $< \mathcal{B}'_2, \mathcal{C}_2(f,k) >$ for arbitrary computable function $k(n)$ if and only if C is of hyperarithmetic degree.*

The proof uses the classical result of Gandy, Kreisel and Tait.

Theorem 3.4 (Gandy, Kreisel and Tait [4])**.** *For any given non-hyperarithmetical set Y, and any non-empty Σ^1_1 set of functions $\mathcal{T}$, there exists $f \in \mathcal{T}$ such that Y is not hyperarithmetical relative to f.*

Solovay [13] using the same kind of forcing but combine with a combinatorial lemma, namely Ellentuck theorem (actually a weaker version), characterize hyperarithmetic degree in a fairly beautiful way as following[1],

Theorem 3.5 (Solovay)**.** *A given set C is of hyperarithmetic degree if and only if for every infinite set X there exists $Y \subseteq X$ such that $C \leq_T Y$.*

4. Cone avoid within partitions

Definition 4.1. Let D_n be the canonical representation of finite set of $2^{<\omega}$.

An enumeration of $T \subseteq 2^{<\omega}$ is a $h : \omega \to \omega$ such that $(\forall n) D_{h(n)} \cap T \neq \emptyset$. Moreover, h is

- k-enumeration iff $(\forall n)|D_{h(n)}| \leq k$;
- non-trivial iff $(\forall n \forall \rho \in D_{h(n)})\ |\rho| = n$;
- strong iff it is a k-enumeration for some $k \in \mathbb{N}$;

In this section we study the partition condition, i.e. the condition of theorem 1.6.

Definition 4.2. $\mathcal{B}_3 = 2^\omega$.
$\mathcal{C}_3(A) = \{G \in 2^\omega : G \subseteq A \vee G \subseteq \overline{A}, |G| = \infty\}$.

We will not only cone avoid a single set within the above condition but a sequence of effective closed set satisfying certain complexity condition.

Definition 4.3. Let $P, Q \subseteq 2^{<\omega}$ be two trees, let $P \bigvee Q = \{\rho \in 2^{<\omega} : \rho \restriction_1^{|\rho|} \in P \wedge \rho(0) = 0 \text{ or } \rho \restriction_1^{|\rho|} \in Q \wedge \rho(0) = 1\}$.

[1]I thank to Liang Yu for telling me the Solovay's paper.

Theorem 4.4. *Let $Q^0, Q^1 \cdots$ be a sequence of trees (not necessarily co-c.e.) such that for any n, $\bigvee_{i=0}^{n} Q^i$ does not admit strong enumeration*[2]*. Then for any set A and any set C such that C does not compute a strong enumeration of any $\bigvee_{i=0}^{n} Q^i$, there exists $G \in \mathcal{C}_3(A)$ such that $C \oplus G$ also does not compute a strong enumeration of any $\bigvee_{i=0}^{n} Q^i$.*

Note that most "natural" trees that does not admit a computable path also does not admit a computable strong enumeration. For example, trees whose strings attain certain complexity, say $Q = \{\sigma \in 2^{<\omega} : (\forall \rho \subset \sigma) K_U(\rho) \geq h(|\rho|)\}$ where $h : \omega \to \omega$ is an unbounded computable function and U a universal prefix free machine. Another trivial example is non-computable set.

In the following applications let $Q^n = \{\sigma \in 2^{<\omega} : (\forall \rho \subset \sigma) K_U(\rho) \geq \frac{1}{n}|\rho| - c\}$ for some appropriate constant c in order to make sure that $Q^n \neq \emptyset$. Note that if C does not compute a path of any Q^n then it certainly does not compute any 1-random real.

Corollary 4.5. RT^2_2 *does not imply* WWKL_0.

Proof. It is well known that $\mathsf{RT}^2_2 \Leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$ (cf say [1]). For any stable coloring, we could use Theorem 4.4 to add an infinite homogeneous set of that coloring that does not compute any 1-random real. It is also easy to construct a cohesive set (of a given uniformly computable sequence of sets) that does not compute any 1-random real. Therefore, we finally obtain a standard arithmetic model that satisfy $\mathsf{SRT}^2_2 + \mathsf{COH}$ but non of its member compute a 1-random real, i.e. it is not a model of WWKL_0.

□

Corollary 4.6. *(Joe Miller's question) There exists a $f \in DNR$ such that f does not compute any real of positive dimension*

Proof. We need,

Theorem 4.7 (Hirschfeldt et.al [6])**.** *There exists A such that every infinite subset G of A or $\overline{A}$, G computes some $f \in DNR$.*

[2] I thank to Joe Miller for that he proposed a question to me on the Workshop of Reverse Mathematics meeting that reminds me of a "flaw" in one of my paper. $k-$*enumeration* of a co-c.e. tree can be used to compute a $1-$*enumeration* of that tree. My answer to his question during my talk may missed the point.

Fix some A as in theorem 4.7 and apply Theorem 4.4 (with $C = \emptyset$) to get a G computing some $f \in DNR$ while G does not compute any strong enumeration of $\widehat{Q}^m$ for all $m \in \omega$, thus does not compute any 1-random. Clearly f does not compute any 1-random.

□

Kjos-Hanssen once asked that does there exist a 1-random such that every infinite subset of it also compute a 1-random real. We give a negative answer and is yet stronger,

Corollary 4.8. *(Kjos-Hanssen's question) For any sequence of co-c.e. binary trees $Q^0, Q^1 \cdots$ that satisfies the condition of Theorem 4.4 and for any 1-random A there exists an infinite set $G \subseteq A$ such that G does not compute any strong enumeration of $\bigvee_{i=0}^{n} Q^i$. Therefore, for any 1-random real A there exists an infinite set $G \subseteq A$ such that G does not compute any real of positive dimension.*

The proof is by slightly modification of that of Theorem 4.4. And in the proof the trees cone avoided is required to be co-c.e., i.e. not like that of Theorem 4.4.

We also mention a coding result here. Simpson asked that whether for every computable two coloring f and every 1-random X, there exists an infinite homogeneous set G of f such that X is 1-random relative to G.

Theorem 4.9. *There exists some 1-random X and a $0'$ computable set A such that every infinite subset of A or $\overline{A}$, G, we have X is not 1-random relative to G. Therefore there exists some computable two coloring f such that for every infinite homogeneous set of f G, X is not 1-random relative to G.*

Proof. Let X be the leftmost path of $Q_n = \{\sigma \in 2^{<\omega} : (\forall \rho \subset \sigma) K_U(\rho) \geq |\rho| - n\}$. Let n_0 be arbitrary, and set $\{0, 1, \dots n_0\} \subset A$. Let n_1 be sufficiently large such that the leftmost string of 2^{n_0} that has not been enumerated in $\overline{Q_n}$ is just $X \upharpoonright n_0$, set $\{n_0+1, n_0+2, \dots n_1\} \subset \overline{A}$. Let n_2 be large enough such that the leftmost string of 2^{n_1} that has not been enumerated in $\overline{Q_n}$ is just $X \upharpoonright n_1$, set $\{n_1+1, n_1+2, \dots n_2\} \subset A \cdots$

□

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