Countable Borel equivalence relations, recursion theory, and Borel combinatorics

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Most equivalence relations from recursion theory are countable Borel equivalence relations (recursive isomorphism, $\equiv_T$, $\equiv_A$, etc.)

Borel reducibility

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If E and F are Borel equivalence relations, then E is said to be **Borel reducible** to F, noted $E \leq_B F$, if there is a Borel function $f : 2^\omega \to 2^\omega$ such that for all $x, y \in 2^\omega$, we have xEy if and only if $f(x)Ff(y)$.

Such an f induces an injection from $2^\omega/E$ to $2^\omega/F$.

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- $\equiv_T \leq_B \equiv_e$ via the map $x \mapsto x \oplus \overline{x}$.
- $\equiv_T \leq_B \equiv_1$ via the map $x \mapsto x'$. (**Folklore**: $x \equiv_T y$ if and only if x' and y' are recursively isomorphic.)

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- Poly-time equivalence (M.)
- Isomorphism of finitely generated groups (Thomas-Veličković, 1999)
- Conformal equivalence of Riemann surfaces (Hjorth-Kechris, 2000)
- Isomorphism of locally finite connected graphs (Kechris?)

*for these latter examples we must use appropriate representations with countable equivalence classes

A cute application of recursion theory

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Theorem (M., answering Jackson-Kechris-Louveau, 2002)

If E is a universal countable Borel equivalence relation, and B is a Borel E-invariant set, then either $E \restriction B$ is universal, or $E \restriction \overline{B}$ is universal.

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Proof: We may as well assume that E is arithmetic equivalence. Slaman and Steel's proof relativizes to show that arithmetic equivalence restricted to any arithmetic cone is still universal. Finally, either B or $\overline{B}$ must contain an arithmetic cone. $\square$

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For example,

Theorem (M.)

If E is a universal countable Borel equivalence relation, and μ is a Borel probability measure on 2^ω, then there's some E-invariant measure 0 set B such that $E \restriction B$ is still universal

Recursion theoretic equivalences under Borel reducibility

What is the structure of equivalence relations from recursion theory organized under Borel reducibility? We might expect most of them to be universal, reflecting a theme in recursion theory where recursion-theoretic structures are often as rich and complicated as possible.

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Open Question (Hjorth, 2001)

Suppose E is a countable Borel equivalence relation such that $E \supseteq \equiv_1$. Must E be universal?

This question probably has a negative answer. The situation appears to be quite deep, and closely tied to longstanding conjectures about the uniformity of degree invariant constructions, among other things.

Martin's conjecture implies $\equiv_T$ is not universal

Conjecture (Martin, 1978)

Suppose f is a Borel Turing invariant function where $x \equiv_T y$ implies $f(x) \equiv_T f(y)$. Then either there exists a constant $z \in 2^\omega$ such that $f(x) \equiv_T z$ on a Turing cone of x, or there exists an $\alpha < \omega_1$ such that $f(x) \equiv_T x^{(\alpha)}$ on a Turing cone of x.

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Note that Martin's conjecture implies that if f is a Borel Turing invariant function that is injective on Turing degrees, then $f(x) \equiv_T x$ on a Turing cone. Hence, Martin's conjecture implies that that $\equiv_T \sqcup \equiv_T \not\leq_B \equiv_T$.

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Slaman and Steel (1988) have showed that Martin's conjecture is true for f whose Turing invariance is witnessed uniformly.

Another interesting case: recursive isomorphism

Definition

Suppose x and y are elements of 2^ω, or more generally n^ω or ω^ω. Then x and y are said to be **recursively isomorphic** if there is a recursive permutation of the bits of x that yields y.

Note that the identity function witnesses

$$\begin{gathered}\text{recursive isomorphism on } \omega^\omega \\ \vdots \\ \rotatebox{90}{$\leq_B$} \\ \text{recursive isomorphism on } 3^\omega \\ \rotatebox{90}{$\leq_B$} \\ \text{recursive isomorphism on } 2^\omega\end{gathered}$$

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Open Question

Is recursive isomorphism on 2^ω a universal countable Borel equivalence relation?

Why is this a difficult problem?

By Dougherty-Jackson-Kechris (1994), there is a Borel action of $F_2 = \langle a, b \rangle$ on 2^ω whose orbit equivalence relation E_{F_2} is universal. This is generally a convenient starting point for universality proofs.

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Suppose now that we wish to embed E_{F_2} into recursive isomorphism **uniformly**, where for all x:

- The images of x and $a \cdot x$ are recursively isomorphic via $\rho : \omega \to \omega$
- The images of x and $b \cdot x$ are recursively isomorphic via $\sigma : \omega \to \omega$

and ρ and σ are independent of x.

Why is this a difficult problem?

We are essentially stuck with the following coding method. Let $\{w_i\}_{i\in\omega}$ be a recursive listing of the words of F_2.

Given any $f : 2^\omega \to 2^\omega$, the function $\hat{f}(x) = \bigoplus_{i\in\omega} f(w_i \cdot x)$ has the property that if $x E_{F_2} y$, then $\hat{f}(x)$ and $\hat{f}(y)$ are recursively isomorphic. Our task would be to find a Borel f so that the converse is also true.

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Problem: Controlling the join of ω many reals is very difficult.

Previous results

Theorem (Dougherty-Kechris, 1991)

Recursive isomorphism on ω^ω is a universal countable Borel equivalence relation

Theorem (Andretta-Camerlo-Hjorth, 2001)

Recursive isomorphism on 5^ω is a universal countable Borel equivalence relation

Progress on this question

Theorem (M.)

Recursive isomorphism on 3^ω is a universal countable Borel equivalence relation

Whether recursive isomorphism on 2^ω is universal remains open. However, we're able to give a concise explanation of the difference between 2 and 3.

- The reason the proof doesn't generalize to recursive isomorphism on 2^ω is a family of graphs that can be 3-colored, but can't be 2-colored.

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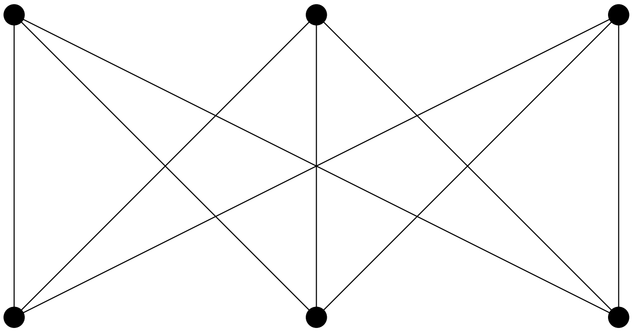
Whether recursive isomorphism on 2^ω is universal remains open. However, we're able to give a concise explanation of the difference between 2 and 3.

- The reason the proof doesn't generalize to recursive isomorphism on 2^ω is a family of graphs that can be 3-colored, but can't be 2-colored.
- Whether recursive isomorphism on 2^ω is universal seems to be related to a problem in Borel combinatorics.

Some basic notions in combinatorics

A graph on X is a symmetric irreflexive relation on X.

An n-**regular** graph is a graph where each vertex has degree n. A **bipartite** graph is a graph whose vertices can be partitioned into two disjoint sets U and V where no two vertices in U are adjacent, and no two vertices in V are adjacent. The graph drawn below is bipartite 3-regular.



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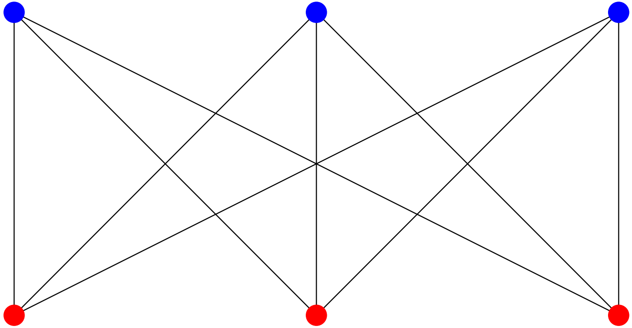
A graph on X is a symmetric irreflexive relation on X.

A **coloring** of a graph is a function f on the vertices of the graph so that if x and y are neighbors, then $f(x) \neq f(y)$. If the range of f is n, then we say f is an n-coloring.



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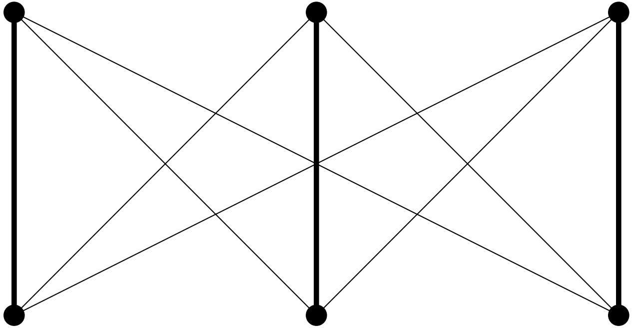
A **matching** of a graph is a subset M of its edges so that each vertex is incident to exactly one edge in M.



Borel combinatorics

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Definition

A **Borel graph** is a graph G whose vertices are the elements of 2^ω, where the edge relation has a Borel definition.

A **Borel coloring** of a Borel graph G with n colors is a Borel function $c : 2^\omega \to n$ that colors G.

Examples of Borel combinatorics

Classical Theorem (Brooks, 1941)

If G is a graph where each vertex has degree less than or equal to d, then there is a coloring of G with $d+1$ colors

Borel Analogue (Kechris-Solecki-Todorčević, 1999)

If G is a Borel graph where each vertex has degree less than or equal to d, then there is a Borel coloring of G with $d+1$ colors.

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Classical Theorem (König, 1916)

Every bipartite n-regular graph has a perfect matching.

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There is a Borel bipartite 2-regular graph with no Borel perfect matching.

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Open

Does every Borel bipartite 3-regular graph have a Borel perfect matching?

The combinatorics of the universality of recursive isomorphism

Suppose $\{G_i\}_{i \in \omega}$ is a sequence of ω many Borel graphs on 2^ω. If $\{c_i\}_{i \in \omega}$ is a sequence of colorings where c_i colors G_i, say that a point $x \in 2^\omega$ is **monochromatic** if it's assigned the same color by all the c_i.

Open Question (*)

Suppose $\{G_i\}_{i \in \omega}$ is a countable sequence of 2-regular Borel graphs. Must there be a countable sequence $\{c_i\}_{i \in \omega}$ of Borel 3-colorings of the G_i with no monochromatic points?

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If this question has an affirmative answer, then recursive isomorphism is universal.

An equivalence

Note that $\equiv_m \leq_B \equiv_1$ via the function $x \mapsto \oplus_{i\in\omega} x$.

Theorem (M.)

(*) has an affirmative answer iff many-1 equivalence is a uniformly universal countable Borel equivalence relation.

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Suppose $\{\phi_i\}_{i \in \omega}$ is a countable collection of partial Borel functions on 2^ω that is closed under composition and includes the identity function. Let $E_{\{\phi_i\}}$ be the associated equivalence relation where $x E_{\{\phi_i\}} y$ iff there exists an i and j such that $\phi_i(x) = y$ and $\phi_j(y) = x$.

Open Question (after Montalbán-Reimann-Slaman)

Suppose $E_{\{\phi_i\}}$ is a universal countable Borel equivalence relation. Must it be uniformly universal?

More about (*)

- If (*) has a negative answer, this would resolve several open questions in Borel combinatorics in the negative. For instance, this would resolve the open question about Borel perfect matchings mentioned earlier.

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- This implies that one can't use pure measure or category arguments to show that recursive isomorphism isn't universal.
- Most negative results in Borel combinatorics use measure or category arguments. A negative answer to (*) would be very interesting since it can't use such techniques.

Where does (*) come from?

It comes from an ω-length construction. At each stage, we obtain directed graphs consisting of odd length directed cycles. We need to assign 0 or 1 to each vertex. Each time this value changes when we move from a vertex to the next vertex, this corresponds to a real on which we've diagonalized. However, since the cycle has odd length, we can't diagonalize everywhere.

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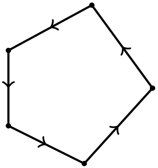

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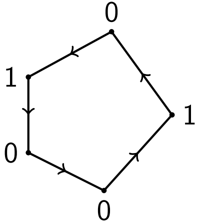

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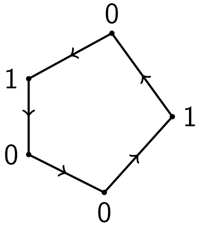

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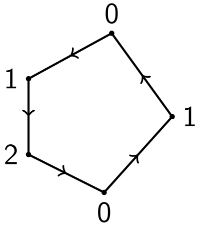

Where does (*) come from?

We get ω many graphs like this, with which we try to diagonalize everywhere. However, if at a single point we use “2” in every coloring, this corresponds to a situation where we never diagonalize on that real.

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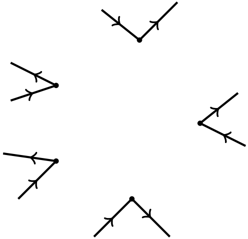

There seems to be a large potential for productive interaction between global problems in recursion theory, countable Borel equivalence relations, and Borel combinatorics.