The topology of ultrafilters as subspaces of 2^ω

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All ultrafilters are non-principal and on ω.
By identifying a subset of ω with an element of 2^ω in the obvious way, we can view any ultrafilter $\mathcal{U}$ as a subspace of 2^ω.

Proposition (folklore)

There are $2^\mathfrak{c}$ non-homeomorphic ultrafilters.

Proof.

Using Lavrentiev's lemma, one sees that the homeomorphism classes have size $\mathfrak{c}$. So there must be $2^\mathfrak{c}$ of them. □

The above proof is a cardinality argument: it is not 'honest' in the sense of Van Douwen. ☹
It would be desirable to get 'quotable' topological properties that distinguish ultrafilters up to homeomorphism.

Similar investigations have been carried out for filters: a delicate interplay emerged between Baire property and Lebesgue measurability. However, these matters are trivial for ultrafilters. Notice that that $2^\omega = \mathcal{U} \sqcup c[\mathcal{U}]$, where $c : 2^\omega \longrightarrow 2^\omega$ is the complement homeomorphism. (So $\mathcal{J} = c[\mathcal{U}]$ is the dual ideal.)

Proposition (folklore)

Every ultrafilter $\mathcal{U} \subseteq 2^\omega$ has the following properties.

- *$\mathcal{U}$ is non-meager and non-comeager.*
- *$\mathcal{U}$ does not have the Baire property.*
- *$\mathcal{U}$ is not Lebesgue measurable.*
- *$\mathcal{U}$ is not analytic and not co-analytic.*
- *$\mathcal{U}$ is a Baire space.*
- *$\mathcal{U}$ is a topological group (hence a homogeneous space).*

The distinguishing properties

From now on, all spaces are separable and metrizable.
Recall the following definitions.

Definition

- A space X is *completely Baire* if every closed subspace of X is a Baire space.
- A space X is *countable dense homogeneous* if for every pair (D, E) of countable dense subsets of X there exists a homeomorphism $h : X \longrightarrow X$ such that $h[D] = E$.
- Given a space X, a subset A of X has the *perfect set property* if A is countable or A contains a homeomorphic copy of 2^ω.

Main results

Theorem

Assume MA(countable). Let P be one of the following topological properties.

- *P = being completely Baire.* ☢
- *P = countable dense homogeneity.*
- *P = every closed subset has the perfect set property.*

Then there exist ultrafilters $\mathcal{U}, \mathcal{V} \subseteq 2^\omega$ *such that* $\mathcal{U}$ *has property P and* $\mathcal{V}$ *does not have property P.* ☺

Question

Can the assumption of MA(countable) be dropped?

Kunen's closed embedding trick

Theorem (Kunen, private communication)

Let C be a zero-dimensional space. Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ with a closed subspace homeomorphic to C.

By choosing $C = \mathbb{Q}$ or $C =$ a Bernstein set one obtains the following corollaries.

Corollary

There exists an ultrafilter $\mathcal{V} \subseteq 2^\omega$ that is not completely Baire.

Corollary

There exists an ultrafilter $\mathcal{V} \subseteq 2^\omega$ with a closed subset that does not have the perfect set property.

Proof of Kunen's trick

Lemma (folklore)

There exists a perfect set $P \subseteq 2^\omega$ such that P is an independent family: that is, every word

$$x_1 \cap \cdots \cap x_m \cap \omega \setminus y_1 \cap \cdots \cap \omega \setminus y_n \text{ is infinite,}$$

where $x_1, \ldots, x_m, y_1, \ldots, y_n \in P$ are distinct.

Let C be the space you want to embed in $\mathcal{V}$ as a closed subset. Since $P \cong 2^\omega$, assume $C \subseteq P$. Now simply define

$$\mathcal{G} = C \cup \{\omega \setminus x : x \in P \setminus C\}.$$

Notice that $\mathcal{G}$ has the finite intersection property because P is independent. Any ultrafilter $\mathcal{V} \supseteq \mathcal{G}$ will intersect P exactly on C.

An ultrafilter that is not countable dense homogeneous

We will use Sierpiński's technique for killing homeomorphisms.

Lemma

Assume MA(countable). Fix D_1 and D_2 disjoint countable dense subsets of 2^ω such that $\mathcal{D} = D_1 \cup D_2$ is an independent family. Then there exists $\mathcal{A} \supseteq \mathcal{D}$ satisfying the following conditions.

- *$\mathcal{A}$ is an independent family.*
- *If $G \supseteq \mathcal{D}$ is a G_δ subset of 2^ω and $f : G \longrightarrow G$ is a homeomorphism such that $f[D_1] = D_2$, then there exists $x \in G$ such that $\{x, \omega \setminus f(x)\} \subseteq \mathcal{A}$.*

In the end, let $\mathcal{V}$ be any ultrafilter extending $\mathcal{A}$.

Enumerate as $\{f_\eta : \eta \in \mathfrak{c}\}$ all such homeomorphisms.
We will construct an increasing sequence of independent families $\mathcal{A}_\xi$ for $\xi \in \mathfrak{c}$. Set $\mathcal{A}_0 = \mathcal{D}$ and take unions at limit stages.
We will take care of f_η at stage $\xi = \eta + 1$, using $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.
List as $\{w_\alpha : \alpha \in \kappa\}$ all the words in $\mathcal{A}_\eta$.
It is easy to check that, for any fixed $n \in \omega$, $\alpha \in \kappa$ and $\varepsilon_1, \varepsilon_2 \in 2$,

$$W_{\alpha,n,\varepsilon_1,\varepsilon_2} = \{x \in G_\eta : |w_\alpha \cap x^{\varepsilon_1} \cap f_\eta(x)^{\varepsilon_2}| \geq n\}$$

is open dense in G_η, so comeager in 2^ω.
So pick x in the intersection of every $W_{\alpha,n,\varepsilon_1,\varepsilon_2}$.

An aside: the separation property

The following property, among Baire spaces, is a weakening of countable dense homogeneity.

Definition (Van Mill, 2009)

A space X has the *separation property* if, given any $A, B \subseteq X$ such that A is meager and B is countable, there exists an homeomorphism $h : X \longrightarrow X$ such that $h[A] \cap B = \varnothing$.

Theorem (Van Mill, 2009)

Every Baire topological group has the separation property.

Corollary

Every ultrafilter $\mathcal{U} \subseteq 2^\omega$ has the separation property.

A countable dense homogeneous ultrafilter

Any ultrafilter $\mathcal{U}$ is homeomorphic to its dual maximal ideal $\mathcal{J}$. So, for notational convenience, we will construct an increasing sequence of ideals $\mathcal{I}_\xi$, for $\xi \in \mathfrak{c}$. In the end, let $\mathcal{J}$ be any maximal ideal extending $\bigcup_{\xi \in \mathfrak{c}} \mathcal{I}_\xi$.
The idea is to use the following lemma.

Lemma

Let $f : 2^\omega \longrightarrow 2^\omega$ be a homeomorphism. Fix a maximal ideal $\mathcal{J} \subseteq 2^\omega$ and a countable dense subset D of $\mathcal{J}$. Then f restricts to a homeomorphism of $\mathcal{J}$ iff $cl(\{d + f(d) : d \in D\}) \subseteq \mathcal{J}$.

Enumerate as $\{(D_\eta, E_\eta) : \eta \in \mathfrak{c}\}$ all pairs of countable dense subsets of 2^ω. At stage $\xi = \eta + 1$, make sure that either

- $\omega \setminus x \in \mathcal{I}_\xi$ for some $x \in D_\eta \cup E_\eta$, or
- there exists an homeomorphism $f : 2^\omega \longrightarrow 2^\omega$ and $x \in \mathcal{I}_\xi$ such that $f[D_\eta] = E_\eta$ and $\{d + f(d) : d \in D_\eta\} \subseteq x\downarrow$.

To construct $f : 2^\omega \longrightarrow 2^\omega$ and x, use MA(countable) on the poset $\mathbb{P}$ consisting of all triples $p = (s, g, \pi) = (s_p, g_p, \pi_p)$ such that, for some $n = n_p \in \omega$, the following conditions hold.

- $s : n \longrightarrow 2$.
- g is a bijection between a finite subset of D and a finite subset of E.
- π is a permutation of ${}^n 2$.
- $(t + \pi(t))(i) = 1$ implies $s(i) = 1$ for every $t \in {}^n 2$ and $i \in n$.
- $\pi(d \restriction n) = g(d) \restriction n$ for every $d \in \mathrm{dom}(g)$.

Order $\mathbb{P}$ by declaring $q \leq p$ if the following conditions hold.

- $s_q \supseteq s_p$.
- $g_q \supseteq g_p$.
- $\pi_q(t) \restriction n_p = \pi_p(t \restriction n_p)$ for all $t \in {}^{n_q} 2$.

An ultrafilter $\mathcal{U}$ such that $A \cap \mathcal{U}$ has the perfect set property whenever A is analytic

Recall that a play of the *strong Choquet game* on a topological space $(X, \mathcal{T})$ is of the form

$$\begin{array}{c|cccccc} \text{I} & (q_0, U_0) & & (q_1, U_1) & & \cdots \\ \hline \text{II} & & V_0 & & V_1 & \cdots, \end{array}$$

where $U_n, V_n \in \mathcal{T}$ are such that $q_n \in V_n \subseteq U_n$ and $U_{n+1} \subseteq V_n$ for every $n \in \omega$.

Player II wins if $\bigcap_{n \in \omega} U_n \neq \varnothing$.

The topological space $(X, \mathcal{T})$ is *strong Choquet* if II has a winning strategy in the above game.

Define an A-*triple* to be a triple of the form $(\mathcal{T}, A, Q)$ such that the following conditions are satisfied.

- $\mathcal{T}$ is a strong Choquet, second-countable topology on 2^ω that is finer than the standard topology.
- $A \in \mathcal{T}$.
- Q is a non-empty countable subset of A with no isolated points in the subspace topology it inherits from $\mathcal{T}$.

For every analytic A there exists a topology $\mathcal{T}$ as above. Also, such a topology $\mathcal{T}$ necessarily consists only of analytic sets. In particular, we can enumerate all A-triples as $\{(\mathcal{T}_\eta, A_\eta, Q_\eta) : \eta \in \mathfrak{c}\}$, making sure that each A-triple appears cofinally often.

We will construct an increasing sequence of filters $\mathcal{F}_\xi$, for $\xi \in \mathfrak{c}$.
Enumerate as $\{z_\eta : \eta \in \mathfrak{c}\}$ all subsets of ω.
At stage $\xi = \eta + 1$, make sure that the following conditions hold.

- Either $z_\eta \in \mathcal{F}_\xi$ or $\omega \setminus z_\eta \in \mathcal{F}_\xi$.
- If $Q_\eta \subseteq \mathcal{F}_\eta$ then there exists $x \in \mathcal{F}_\xi$ such that $x\!\uparrow \cap A_\eta$ contains a perfect subset.

Let $\mathcal{U} = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}_\xi$. If $A \cap \mathcal{U}$ is uncountable for some analytic A then it must have an uncountable subset S with no isolated points. Hence there exists some $Q \subseteq S$ and $\mathcal{T}$ such that $(\mathcal{T}, A, Q)$ is an A-triple. So we took care of it.

Given an A-triple $(\mathcal{T}, A, Q) = (\mathcal{T}_\eta, A_\eta, Q_\eta)$, construct x by applying MA(countable) to the following poset.
Fix a winning strategy Σ for player II in the strong Choquet game in $(2^\omega, \mathcal{T})$. Also, fix a countable base $\mathcal{B}$ for $(2^\omega, \mathcal{T})$.
Let $\mathbb{P}$ be the countable poset consisting of all functions p such that for some $n = n_p \in \omega$ the following conditions hold.

- $p : {}^{\leq n}2 \longrightarrow Q \times \mathcal{B}$. We will use the notation $p(s) = (q_s^p, U_s^p)$.
- $U_\varnothing^p = A$.
- For every $s, t \in {}^{\leq n}2$, if s and t are incompatible (that is, $s \not\subseteq t$ and $t \not\subseteq s$) then $U_s^p \cap U_t^p = \varnothing$.

- For every $s \in {}^n 2$,

$$\begin{array}{c|ccccccc} \text{I} & (q^p_{s\restriction 0}, U^p_{s\restriction 0}) & & \cdots & (q^p_{s\restriction n}, U^p_{s\restriction n}) & \\ \hline \text{II} & & V^p_{s\restriction 0} & \cdots & & V^p_{s\restriction n} \end{array}$$

  is a partial play of the strong Choquet game in $(2^\omega, \mathcal{T})$, where the open sets $V^p_{s\restriction i}$ played by II are the ones dictated by the strategy Σ.

Order $\mathbb{P}$ by setting $p \leq p'$ whenever $p \supseteq p'$.

The generic tree will naturally yield a perfect set P such that $\mathcal{F}_\eta \cup \{\bigcap P\}$ has the finite intersection property.
So set $x = \bigcap P$.

A question of Hrušák and Zamora Avilés

Hrušák and Zamora Avilés showed that, for a Borel $X \subseteq 2^\omega$, the following conditions are equivalent.

- X^ω is countable dense homogeneous.
- X is a G_δ.

Then they asked whether there exists a non-G_δ subset X of 2^ω such that X^ω is countable dense homogeneous.
The following theorem consistently answers their question.

Theorem

Assume MA(countable). Then there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is countable dense homogeneous.

Extending the perfect set property

Under V=L, there exists a co-analytic subset of 2^ω without the perfect set property. So MA(countable) is not enough to extend the perfect set property to $\mathcal{U} \cap A$ for all co-analytic A.

Theorem

Assume the consistency of a Mahlo cardinal. Then it is consistent that there exists an ultrafilter $\mathcal{U} \subseteq 2^\omega$ such $A \cap \mathcal{U}$ has the perfect set property for all $A \in \mathcal{P}(2^\omega) \cap L(\mathbb{R})$.

At least an inaccessible is needed for the above theorem.

Question

Does the Levy collapse of an inaccessible κ to ω_1 force such an ultrafilter?

P-point = completely Baire

Theorem (Marciszewski, 1998)

Let $\mathcal{F} \subseteq 2^\omega$ be a filter. Then $\mathcal{F}$ is completely Baire if and only if it is a non-meager P-filter.

So the problem of completely Baire ultrafilters has been completely solved already, by the following well-known results.

Proposition (folklore)

There exist non-P-points.

Theorem (W. Rudin, 1956, plus folklore)

Assume MA(countable). Then there exist P-points.

Theorem (Shelah, 1982)

It is consistent that there are no P-points.

P-points and the perfect set property

We constructed the following examples.

	P-point	non-P-point
psp	✓	?
non-psp	?	✓

Question

For an ultrafilter $\mathcal{U} \subseteq 2^\omega$, is being a P-point equivalent to $\mathcal{U} \cap A$ having the perfect set property whenever $A \subseteq 2^\omega$ is analytic?

Theorem

Let $\mathcal{U}$ be a P_{ω_2}-point. Then $A \cap \mathcal{U}$ has the perfect set property whenever $A \subseteq 2^\omega$ is such that every closed subset of A has the perfect set property. (For example, whenever A is analytic).

Non-meager P-filters are countable dense homogeneous

Theorem (Hernández-Gutiérrez and Hrušák, preprint)

If $\mathcal{F} \subseteq 2^\omega$ is a non-meager P-filter then both $\mathcal{F}$ and $\mathcal{F}^\omega$ are countable dense homogeneous.

However, their result does not make our proofs useless, ☺ because they can easily be modified to obtain the following.

Theorem

Assume MA(countable). Then there exists a non-P-point $\mathcal{U} \subseteq 2^\omega$ that is countable dense homogeneous.

Theorem

Assume MA(countable). Then there exists a non-P-point $\mathcal{U} \subseteq 2^\omega$ such that $\mathcal{U}^\omega$ is countable dense homogeneous.

Directions for future research

The result of Hernández-Gutiérrez and Hrušák suggests the two following problems.

Problem

Find a combinatorial characterization of countable dense homogeneous ultrafilters.

For example, is weak P-point the same as countable dense homogeneous? What about ω_1-OK ultrafilters?

Question

Can Shelah's proof of the consistency of no P-points be modified to yield a model with no countable dense homogeneous ultrafilters?