Lebesgue density and cupping with K-trivial sets

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Effective randomness

There are several notions of "effective randomness". They are usually defined by isolating a *countable* collection of nice measure zero sets $\{C_0, C_1, \ldots\}$.

Then:

Definition

$X \in 2^\omega$ is *random* if $X \notin \bigcup_n C_n$.

The most important example was given by Martin-Löf in 1966. We give a definition due to Solovay:

Definition

A *Solovay test* is a computable sequence $\{\sigma_n\}_{n\in\omega}$ of elements of $2^{<\omega}$ (finite binary strings) such that $\sum_n 2^{-|\sigma_n|} < \infty$.

The test *covers* $X \in 2^\omega$ if X has infinitely many prefixes in $\{\sigma_n\}_{n\in\omega}$.

$X \in 2^\omega$ is *Martin-Löf random* if no Solovay test covers it.

Martin-Löf randomness

Why is Martin-Löf randomness a good notion?

1. It has nice properties
   - Satisfies all reasonable statistical tests of randomness
   - Plays well with computability-theoretic notions
2. It has several natural characterizations

Let K denote *prefix-free (Kolmogorov) complexity*. Intuitively, $K(\sigma)$ is the length of the shortest (binary, self-delimiting) description of σ.

Theorem (Schnorr)

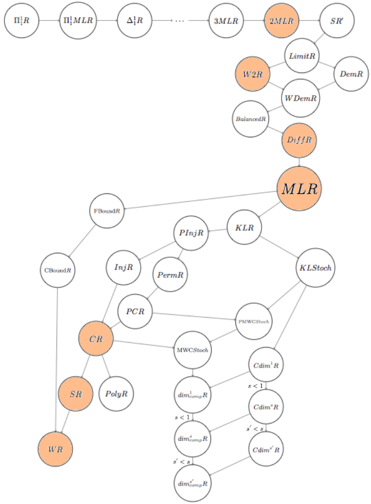
X is Martin-Löf random iff $K(X \upharpoonright n) \geqslant n - O(1)$.

In other words, a sequence is Martin-Löf random iff its initial segments are *incompressible*.

Martin-Löf random sequences can also be characterized as *unpredictable*; it is hard to win money betting on the bits of a Martin-Löf random.

Other randomness notions

2-randomness
⇓
weak 2-randomness
⇓
difference randomness
⇓
Martin-Löf randomness
(1-randomness)
⇓
Computable randomness
⇓
Schnorr randomness
⇓
Kurtz randomness
(weak 1-randomness)



Randomness Zoo (Antoine Taveneaux)

A template for randomness and analysis

Many results in analysis and related fields look like this:

Classical Theorem

Given a mimsy borogove M, almost every x is frabjous for M.

There are only countably many effective borogoves, so

Corollary

Almost every x is frabjous for *every* effective mimsy borogove.

Thus a sufficiently strong randomness notion will guarantee being frabjous for every effective mimsy borogove.

Question

How much randomness is necessary?

Ideally, we get a characterization of a natural randomness notion:

Ideal Effectivization of the Classical Theorem

x is *Alice* random iff x is frabjous for every effective mimsy borogove.

Randomness and analysis (examples)

Examples will clarify:

Classical Theorem

Every function $f\colon [0,1] \to \mathbb{R}$ of bounded variation is differentiable at almost every $x \in [0,1]$.

Ideal Effectivization (Demuth 1975)

A real $x \in [0,1]$ is Martin-Löf random iff every computable $f\colon [0,1] \to \mathbb{R}$ of bounded variation is differentiable at x.

Classical Theorem (a special case of the previous example)

Every monotonic function $f\colon [0,1] \to \mathbb{R}$ is differentiable at almost every $x \in [0,1]$.

Ideal Effectivization (Brattka, M., Nies)

A real $x \in [0,1]$ is computably random iff every monotonic computable $f\colon [0,1] \to \mathbb{R}$ is differentiable at x.

Randomness and analysis (more examples)

An effectivization of a form of the Lebesgue differentiation theorem (also related to the previous examples):

Theorem (Rute; Pathak, Rojas and Simpson)

A real $x \in [0,1]$ is Schnorr random iff the integral of an $\mathcal{L}_1$-computable $f\colon [0,1] \to \mathbb{R}$ must be differentiable at x.

An effectivization of (a form of) Birkhoff's Ergodic Theorem:

Theorem (Franklin, Greenberg, M., Ng; Bienvenu, Day, Hoyrup, Mezhirov, Shen)

Let M be a computable probability space, and let $T\colon M \to M$ be a computable ergodic map. Then a point $x \in M$ is Martin-Löf random iff for every Π^0_1 class $P \subseteq M$,

$$\lim_{n\to\infty} \frac{\#\left\{i < n\colon T^i(x) \in P\right\}}{n} = \mu(P).$$

There are a handful of other examples.

Lebesgue density

We would like to do the same kind of analysis for (a form of) the Lebesgue Density Theorem.

Definition

Let $C \in 2^\omega$ be measurable. The *lower density of* $X \in C$ is

$$\rho(X \mid C) = \liminf_n \frac{\mu([X \upharpoonright n] \cap C)}{2^{-n}}.$$

Here, μ is the standard Lebesgue measure on Cantor space and $[\sigma] = \{Z \in 2^\omega \mid \sigma \prec Z\}$, so $\mu([X \upharpoonright n]) = 2^{-n}$.

Lebesgue Density Theorem

If $C \in 2^\omega$ is measurable, then $\rho(X \mid C) = 1$ for almost every $X \in C$.

We want to understand the density points of Π^0_1 classes.

Lebesgue density

We want to understand the density points of Π^0_1 classes.

Question

For which X is it the case that $\rho(X \mid C) = 1$ for every Π^0_1 class C containing X.

Note. Every 1-generic has this property. So this is not going to characterize a natural randomness class.

Theorem (Bienvenu, Hölzl, M., Nies)

Assume that X is Martin-Löf random. Then $X \geqslant_T \emptyset'$ iff there is a Π^0_1 class C containing X such that $\rho(X \mid C) = 0$.

Notes:

- We have not been able to extend this to $\rho(X \mid C) < 1$.
- If $\mu(C)$ is computable, then by the effectivization of the Lebesgue differentiation theorem, every *Schnorr random* in C is a density point of C.

Difference randomness

Theorem (Bienvenu, Hölzl, M., Nies)

Assume that X is Martin-Löf random. Then $X \geqslant_T \emptyset'$ iff there is a Π^0_1 class C containing X such that $\rho(X \mid C) = 0$.

The contrapositive lets us characterize the Martin-Löf randoms that do not compute $\emptyset'$ (which will be very useful!). It is not the first such characterization.

Definition (Franklin and Ng)

A *(Solovay-rian) difference test* is a Π^0_1 class C and a computable sequence $\{\sigma_n\}_{n\in\omega}$ of elements of $2^{<\omega}$ such that $\sum_n \mu([\sigma_n] \cap C) < \infty$.

The test *covers* $X \in C$ if X has infinitely many prefixes in $\{\sigma_n\}_{n\in\omega}$.

$X \in 2^\omega$ is *difference random* if no difference test covers it.

Essentially, a difference test is just a Solovay test (or usually, a Martin-Löf test) inside a Π^0_1 class.

Difference randomness

Theorem (Franklin and Ng)

X is difference random iff X is Martin-Löf random and $X \not\geq_T \emptyset'$.

It can be shown:

Lemma

Let C be a Π^0_1 class and $X \in C$ Martin-Löf random. TFAE:

1. $\rho(X \mid C) = 0$.
2. There is a computable sequence $\{\sigma_n\}_{n\in\omega}$ such that C and $\{\sigma_n\}_{n\in\omega}$ form a difference test.

From which our result follows immediately:

Theorem (Bienvenu, Hölzl, M., Nies)

Assume that X is Martin-Löf random. Then $X \geqslant_T \emptyset'$ iff there is a Π^0_1 class C containing X such that $\rho(X \mid C) = 0$.

K-triviality

The previous result has an application to K-triviality.

Theorem (variously Nies, Hirschfeldt, Stephan, ...)

The following are equivalent for $A \in 2^\omega$:

1. $K(A \upharpoonright n) \leqslant K(n) + O(1)$ (A is K-*trivial*).
2. Every Martin-Löf random X is Martin-Löf random relative to A (A is *low for random*).
3. There is an $X \geqslant_T A$ that is Martin-Löf random relative to A.

$$\vdots$$

17. For every A-c.e. set $F \subseteq 2^{<\omega}$ such that $\sum_{\sigma \in F} 2^{-|\sigma|} < \infty$, there is a c.e. set $G \supseteq F$ such that $\sum_{\sigma \in G} 2^{-|\sigma|} < \infty$.

Other Facts

- [Solovay 1975] There is a non-computable K-trivial set.
- [Chaitin] Every K-trivial is $\leqslant_T \emptyset'$.
- [Nies, Hirschfeldt] Every K-trivial is low ($A' \leqslant_T \emptyset'$).

(Weakly) ML-cupping

Definition (Kučera 2004)

$A \in 2^\omega$ is *weakly ML-cuppable* if there is a Martin-Löf random sequence $X \ngeqslant_T \emptyset'$ such that $A \oplus X \geqslant_T \emptyset'$. If one can choose $X <_T \emptyset'$, then A is *ML-cuppable*.

Question (Kučera)

Can the K-trivial sets be characterized as either

1. not weakly ML-cuppable, or
2. $\leqslant_T \emptyset'$ and not ML-cuppable?

Compare this to:

Theorem (Posner and Robinson)

For every $A >_T \emptyset$ there is a 1-generic X such that $A \oplus X \geqslant_T \emptyset'$. If $A \leqslant_T \emptyset'$, then also $X \leqslant_T \emptyset'$.

(Weakly) ML-cupping

Question (Kučera 2004)

Can the K-trivial sets be characterized as either

1. not weakly ML-cuppable, or
2. $\leqslant_T \emptyset'$ and not ML-cuppable?

Answer (Day and M.)

Yes, both.

Partial results

- If $A \leqslant_T \emptyset'$ and not K-trivial, it is weakly ML-cuppable (by Ω^A).
- If A is low and not K-trivial, then it is ML-cuppable (by Ω^A). (Also any A that can be shown to compute a low non-K-tivial.)
- [Nies] There is a non-computable K-trivial c.e. set that is not weakly ML-cuppable.

Answering Kučera's question

Theorem (Day and M.)

If A is not K-trivial, then it is weakly ML-cuppable (i.e., there is a Martin-Löf random sequence $X \not\geqslant_T \emptyset'$ such that $A \oplus X \geqslant_T \emptyset'$). If $A <_T \emptyset'$ is not K-trivial, then it is ML-cuppable (i.e., we can take $X \leqslant_T \emptyset'$ too).

These are proved by straightforward constructions.

Idea. Given A, we (force with positive measure Π^0_1 classes to) construct a Martin-Löf random X that is not Martin-Löf random relative to A. We code the settling-time function for $\emptyset'$ into $A \oplus X$ by alternately making X look A-random for long stretches and then dropping $K^A(X \upharpoonright n)$ for some n.

It is the other direction I want to focus on.

Theorem (Day and M.)

If A is K-trivial, then it is not weakly ML-cuppable.

This involves the work on Lebesgue density and Π^0_1 classes.

Answering Kučera's question

Theorem (Day and M.)

If A is K-trivial, then it is not weakly ML-cuppable.

Proof.

Let A be K-trivial, X Martin-Löf random, and $A \oplus X \geqslant_T \emptyset'$. We will show that $X \geqslant_T \emptyset'$.

Because A is K-trivial it is low ($\emptyset' \geqslant_T A'$), hence $A \oplus X \geqslant_T A'$. It is also low for random, so X is Martin-Löf random relative to A. Therefore, by the Bienvenu et al. result relativized to A, there is a $\Pi^0_1[A]$ class C containing X such that $\rho(X \mid C) = 0$.

Let $F \subseteq 2^{<\omega}$ be an A-c.e. set such that

$$2^\omega \setminus C = [F] = \bigcup_{\sigma \in F} [\sigma].$$

We may assume that F is prefix-free, hence $\sum_{\sigma \in F} 2^{-|\sigma|} \leqslant 1 < \infty$.

⋮

Answering Kučera's question

Theorem (Day and M.)

If A is K-trivial, then it is not weakly ML-cuppable.

Proof continued.

$\vdots$

By characterization 17 of K-triviality, there is a c.e. set $G \supseteq F$ such that $\sum_{\sigma \in G} 2^{-|\sigma|} < \infty$.

This G is a *Solovay test*. Because X is Martin-Löf random, there are only finitely many $\sigma \in G$ such that $\sigma \prec X$. No such σ is in F, so without loss of generality, we may assume that no such σ is in G.

Consider the Π^0_1 class $D = 2^\omega \smallsetminus [G]$. Note that $X \in D$. Also, $D \subseteq C$, so $\rho(X \mid D) = 0$. Therefore, by the Bienvenu et al. result, $X \geqslant_T \emptyset'$.

In other words, X does not witness the weak ML-cuppability of A. □

Kučera's question answered

Theorem (various)

The following are equivalent for $A \in 2^\omega$:

1. $K(A \upharpoonright n) \leqslant K(n) + O(1)$ (A is K-*trivial*).

$\vdots$

18. A is not weakly ML-cuppable.
19. $A \leqslant_T \emptyset'$ and A is not ML-cuppable.

These are the first characterizations of K-triviality in term of their interactions in the Turing degrees with the degrees of ML-randoms.

By improving the cupping direction, we can even remove any mention of $\emptyset'$.

20. There is a $D >_T \emptyset$ such that if X is Martin-Löf random and $A \oplus X \geqslant_T D$, then $X \geqslant_T D$. (also with Adam Day)

Lebesgue density revisited

Suppose that C is a Π^0_1 class and $X \in C$.

We know that if X is difference random, then $\rho(X \mid C) > 0$. But we *wanted* to characterize the X such that $\rho(X \mid C) = 1$.

Definition

Call $X \in 2^\omega$ a *non-density point* if there is a Π^0_1 class C such that $X \in C$ and $\rho(X \mid C) < 1$.

Lemma (Bienvenu, Hölzl, M., Nies)

Assume that X is a Martin-Löf random non-density point. Then X computes a function f (witnessing its non-density) such that for every A either:

- f dominates every A-computable function, or
- X is not Martin-Löf random relative to A.

Lebesgue density revisited

Taking $A = \emptyset$, this shows that a Martin-Löf random non-density point computes a function that dominates every computable function. In other words:

Theorem (Bienvenu, Hölzl, M., Nies)

If X is a Martin-Löf random non-density point, then X is high ($X' \geqslant_T \emptyset''$).

In fact, X is Martin-Löf random relative to almost every A, so f must dominate every A-computable function for almost every A.

Theorem (Bienvenu, Hölzl, M., Nies)

If X is a Martin-Löf random non-density point, then X is *(uniformly) almost everywhere dominating*.

So for Martin-Löf random sequences:

$$\text{not a.e.d} \implies \text{density point for } \Pi^0_1 \text{ classes} \implies \text{not} \geqslant_T \emptyset'.$$

Lebesgue density revisited

If A is a computably enumerable set, then A computes a function g (its settling-time function) such that every function dominating g computes A. Therefore:

Lemma

If X is a Martin-Löf random non-density point and A is c.e., then either $X \geqslant_T A$ or X is not Martin-Löf random relative to A.

So if A is K-trivial (hence low for random) and c.e., then X must compute A! But every K-trivial is bounded by a c.e. K-trivial (Nies), so:

Theorem (Greenberg, Nies, Turetsky??)

If X is a Martin-Löf random non-density point, then X computes *every* K-trivial.

This is related to another open question about the K-trivial sets.

ML-covering

Question (Stephan 2004)

If A is K-trivial, must there be a Martin-Löf random $X \geqslant_T A$ such that $X \not\geqslant_T \emptyset'$?

Together with the following result, this would give a new characterization of the c.e. K-trivial sets:

Theorem (Hirschfeldt, Nies, Stephan)

If A is c.e., X is Martin-Löf random, $X \geqslant_T A$ but $X \not\geqslant_T \emptyset'$, then A is K-trivial.

But now we see that this is connected to Lebesgue density:

Fact

If there a Martin-Löf random non-density point $X \not\geqslant_T \emptyset'$, then the question has a positive answer: every K-trivial is below a Martin-Löf random that does not compute $\emptyset'$ (because they are all below X!).

Thank You!