The complexity within well-partial-orderings

Antonio Montalbán

–

University of Chicago

Madison, March 2012

1. Background on WQOs

2. WQOs in Proof Theory
   - Kruskal's theorem and the graph-minor theorem
   - Linear orderings and Fraïssé's Conjecture

3. WPOs in Computability Theory

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- countable linear orderings [Laver 71];

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- countable linear orderings [Laver 71];
- finite graphs [Robertson, Seymour].

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- countable linear orderings [Laver 71];
- finite graphs [Robertson, Seymour].

Well-quasi-orderings

Definition: A *well-quasi-ordering (WQO)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- countable linear orderings [Laver 71];
- finite graphs [Robertson, Seymour].

Definition:
A *well-partial-ordering (WPO)*, is a WQO which is a partial ordering.

Well-partial-orders

There are many equivalent characterizations of WPOs:

Well-partial-orders

There are many equivalent characterizations of WPOs:

- $\mathcal{P}$ is well-founded and has no infinite antichains;

Well-partial-orders

There are many equivalent characterizations of WPOs:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leqslant_P f(j)$;

Well-partial-orders

There are many equivalent characterizations of WPOs:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leqslant_P f(j)$;
- every subset of P has a finite set of minimal elements;

Well-partial-orders

There are many equivalent characterizations of WPOs:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leqslant_P f(j)$;
- every subset of P has a finite set of minimal elements;
- all linear extensions of $\mathcal{P}$ are well-orders.

Well-partial-orders

There are many equivalent characterizations of WPOs:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leqslant_P f(j)$;
- every subset of P has a finite set of minimal elements;
- all linear extensions of $\mathcal{P}$ are well-orders.

The reverse mathematics and computability theory of these equivalences was been studied in [Cholak-Marcone-Solomon 04].

Closure properties of WPOs

Closure properties of WPOs

- The sum and disjoint sum of two WPOs are WPO.

Closure properties of WPOs

- The sum and disjoint sum of two WPOs are WPO.
- The product of two WPOs is WPO.

Closure properties of WPOs

- The sum and disjoint sum of two WPOs are WPO.
- The product of two WPOs is WPO.
- Finite strings over a WPO are a WPO (Higman, 1952).

Closure properties of WPOs

- The sum and disjoint sum of two WPOs are WPO.
- The product of two WPOs is WPO.
- Finite strings over a WPO are a WPO (Higman, 1952).
- Finite trees with labels from a WPO are a WPO (Kruskal, 1960).

Closure properties of WPOs

- The sum and disjoint sum of two WPOs are WPO.
- The product of two WPOs is WPO.
- Finite strings over a WPO are a WPO (Higman, 1952).
- Finite trees with labels from a WPO are a WPO (Kruskal, 1960).
- Transfinite sequences with labels from a WPO which use only finitely many labels are a WPO (Nash-Williams, 1965).

Length

Recall: Every linearization of a WPO is well-ordered.

Length

Recall: Every linearization of a WPO is well-ordered.
($\leqslant_L$ is a *linearization* of $(P, \leqslant_P)$ if it's linear and $x \leqslant_P y \Rightarrow x \leqslant_L y$.

Length

Recall: Every linearization of a WPO is well-ordered.
($\leqslant_L$ is a *linearization* of $(P, \leqslant_P)$ if it's linear and $x \leqslant_P y \Rightarrow x \leqslant_L y$.
So, for any $\{x_n\}_{n\in\omega}$, there are $i < j$ with $(x_i \leqslant_P x_j)$, hence $x_i \not> _L x_j$.)

Length

Recall: Every linearization of a WPO is well-ordered.
($\leqslant_L$ is a *linearization* of $(P, \leqslant_P)$ if it's linear and $x \leqslant_P y \Rightarrow x \leqslant_L y$.
So, for any $\{x_n\}_{n\in\omega}$, there are $i < j$ with $(x_i \leqslant_P x_j)$, hence $x_i \not\geqslant_L x_j$.)

Definition: The *length* of $\mathcal{P} = (P, \leqslant_P)$ is
$o(\mathcal{P}) = \sup\{\mathrm{ordType}(W, \leqslant_L) :$ where $\leqslant_L$ is a linearization of $\mathcal{P}\}$.

Length

Recall: Every linearization of a WPO is well-ordered.
($\leqslant_L$ is a *linearization* of $(P, \leqslant_P)$ if it's linear and $x \leqslant_P y \Rightarrow x \leqslant_L y$.
So, for any $\{x_n\}_{n\in\omega}$, there are $i < j$ with $(x_i \leqslant_P x_j)$, hence $x_i \not\geqslant_L x_j$.)

Definition: The *length* of $\mathcal{P} = (P, \leqslant_P)$ is
$o(\mathcal{P}) = \sup\{\mathrm{ordType}(W, \leqslant_L) :$ where $\leqslant_L$ is a linearization of $\mathcal{P}\}$.

Def: $\mathbb{B}\mathrm{ad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1}\rangle \in P^{<\omega} : \forall i < j\ (x_i \not\leqslant_P x_j)\}$,
Note: $\mathcal{P}$ is a WPO $\Leftrightarrow$ $\mathbb{B}\mathrm{ad}(\mathcal{P})$ is well-founded.

Length

Recall: Every linearization of a WPO is well-ordered.
($\leqslant_L$ is a *linearization* of $(P, \leqslant_P)$ if it's linear and $x \leqslant_P y \Rightarrow x \leqslant_L y$.
So, for any $\{x_n\}_{n\in\omega}$, there are $i < j$ with $(x_i \leqslant_P x_j)$, hence $x_i \not\geqslant_L x_j$.)

Definition: The *length* of $\mathcal{P} = (P, \leqslant_P)$ is
$o(\mathcal{P}) = \sup\{\mathrm{ordType}(W, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

Def: $\mathbb{B}\mathrm{ad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1}\rangle \in P^{<\omega} : \forall i < j\ (x_i \not\leqslant_P x_j)\}$,
Note: $\mathcal{P}$ is a WPO $\Leftrightarrow$ $\mathbb{B}\mathrm{ad}(\mathcal{P})$ is well-founded.

Theorem: [De Jongh, Parikh 77] $o(\mathcal{P}) + 1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{P}))$.

1 Background on WQOs

2 WQOs in Proof Theory
- Kruskal's theorem and the graph-minor theorem
- Linear orderings and Fraïssé's Conjecture

3 WPOs in Computability Theory

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$, the Feferman–Schütte ordinal.

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding $: T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$, the Feferman–Schütte ordinal.
(Γ_0 is the the proof-theoretic ordinal of ATR_0.
It's the "least ordinal" that ATR_0 can't prove it's an ordinal.

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by
$T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$, the Feferman–Schütte ordinal.
(Γ_0 is the the proof-theoretic ordinal of ATR_0.
It's the "least ordinal" that ATR_0 can't prove it's an ordinal.

ATR_0 –Arithmetic Transfinite Recursion– is the subsystem of 2nd-order arithmetic that allows the iteration of the Turing jump along any ordinal.)

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$, the Feferman–Schütte ordinal.
(Γ_0 is the the proof-theoretic ordinal of ATR_0.
It's the "least ordinal" that ATR_0 can't prove it's an ordinal.

ATR_0 –Arithmetic Transfinite Recursion– is the subsystem of 2nd-order arithmetic that allows the iteration of the Turing jump along any ordinal.)

Corollary: [Friedman] (RCA_0) Kruskal's theorem $\Rightarrow$ Γ_0 well-ordered.

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $<$ and $g.l.b.$
Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$, the Feferman–Schütte ordinal.
(Γ_0 is the the proof-theoretic ordinal of ATR_0.
It's the "least ordinal" that ATR_0 can't prove it's an ordinal.

ATR_0 –Arithmetic Transfinite Recursion– is the subsystem of 2nd-order arithmetic that allows the iteration of the Turing jump along any ordinal.)

Corollary: [Friedman] (RCA_0) Kruskal's theorem $\Rightarrow \Gamma_0$ well-ordered.
Therefore,

$$ATR_0 \nvdash \text{Kruskal's theorem.}$$

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0:

WKL_0:

ACA_0:

ATR_0:

$\Pi^1_1\text{-}\mathsf{CA}_0$:

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0:

ACA_0:

ATR_0:

Π^1_1-CA_0:

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's lemma + RCA_0

ACA_0:

ATR_0:

Π^1_1-CA_0:

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's lemma + RCA_0

ACA_0: Arithmetic Comprehension + RCA_0
$\Leftrightarrow$ "for every set X, X' exists".

ATR_0:

Π^1_1-CA_0:

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's lemma + RCA_0

ACA_0: Arithmetic Comprehension + RCA_0

$\Leftrightarrow$ "for every set X, X' exists".

ATR_0: Arithmetic Transfinite recursion + ACA_0.

$\Leftrightarrow$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".

Π^1_1-CA_0:

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's lemma + RCA_0

ACA_0: Arithmetic Comprehension + RCA_0

$\Leftrightarrow$ "for every set X, X' exists".

ATR_0: Arithmetic Transfinite recursion + ACA_0.

$\Leftrightarrow$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".

Π^1_1-CA_0: Π^1_1-Comprehension + ACA_0.

$\Leftrightarrow$ "$\forall X$, the hyper-jump of X exists".

The exact reversals

[Friedman] Neither of ATR_0, or Kruskal's theorem implies the other.

The exact reversals

[Friedman] Neither of ATR_0, or Kruskal's theorem implies the other.

Thm: [Rathjen–Weiermann 93] The length of $\mathcal{T}$ is $\theta\Omega^{\omega}$, the Ackerman ordinal. The following are equivalent over RCA_0

- Kruskal's theorem.
- The Π^1_1-reflection principle for Π^1_2-transfinite induction.

The exact reversals

[Friedman] Neither of ATR_0, or Kruskal's theorem implies the other.

Thm: [Rathjen–Weiermann 93] The length of $\mathcal{T}$ is $\theta\Omega^\omega$, the Ackerman ordinal.
The following are equivalent over RCA_0

- Kruskal's theorem.
- The Π^1_1-reflection principle for Π^1_2-transfinite induction.

Thm: [M.–Weiermann 2006] The following are equivalent over RCA_0

- ATR_0
- For every $\mathcal{P}$, *if $\mathcal{P}$ is a WQO, then so is $\mathcal{T}(\mathcal{P})$*,
  where $\mathcal{T}(\mathcal{P})$ is the set of finite trees with labels in $\mathcal{P}$, ordered by
  $T \preceq S$ if $\exists f : T \to S$ which preserves $<$ and increasing on labels.

The minor-graph theorem

Theorem: [Robertson–Seymour]
Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation.
Then $\mathcal{G}$ is a WQO.

The minor-graph theorem

Theorem: [Robertson–Seymour]
Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation.
Then $\mathcal{G}$ is a WQO.

Theorem: [Friedman–Robertson–Seymour] The length of $\mathcal{G}$ is $\geqslant \phi_0(\epsilon_{\Omega_\omega+1})$.

The minor-graph theorem

Theorem: [Robertson–Seymour]
Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation.
Then $\mathcal{G}$ is a WQO.

Theorem: [Friedman–Robertson–Seymour] The length of $\mathcal{G}$ is $\geqslant \phi_0(\epsilon_{\Omega_\omega+1})$.
(where $\phi_0(\epsilon_{\Omega_\omega+1})$, the Takeuti-Feferman-Buchholz ordinal,
is the the proof-theoretic ordinal of Π^1_1-CA_0.

The minor-graph theorem

Theorem: [Robertson–Seymour]
Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation.
Then $\mathcal{G}$ is a WQO.

Theorem: [Friedman–Robertson–Seymour] The length of $\mathcal{G}$ is $\geqslant \phi_0(\epsilon_{\Omega_\omega+1})$.
(where $\phi_0(\epsilon_{\Omega_\omega+1})$, the Takeuti-Feferman-Buchholz ordinal,
is the the proof-theoretic ordinal of Π^1_1-CA_0.

Π^1_1-CA_0 – is the system that allows Π^1_1-comprehension.)

The minor-graph theorem

Theorem: [Robertson–Seymour]
Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation.
Then $\mathcal{G}$ is a WQO.

Theorem: [Friedman–Robertson–Seymour] The length of $\mathcal{G}$ is $\geqslant \phi_0(\epsilon_{\Omega_\omega+1})$.
(where $\phi_0(\epsilon_{\Omega_\omega+1})$, the Takeuti-Feferman-Buchholz ordinal,
is the the proof-theoretic ordinal of Π^1_1-CA_0.
Π^1_1-CA_0 – is the system that allows Π^1_1-comprehension.)

Corollary: [Friedman, Robertson, Seymour]
(RCA_0) The minor-grarph theorem $\Rightarrow \phi_0(\epsilon_{\Omega_\omega+1})$ well-ordered.
Therefore,

$$\Pi^1_1\text{-}\mathsf{CA}_0 \not\vdash \text{minor-graph theorem}.$$

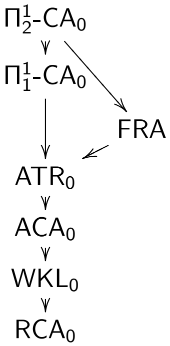
Theorem [Fraïssé's Conjecture '48; Laver '71]
FRA:The countable linear orderings are WQO under embeddablity.

Fraïssé's Conjecture

Theorem [Fraïssé's Conjecture '48; Laver '71]
FRA:The countable linear orderings are WQO under embeddablity.

Theorem[Shore '93]
FRA implies ATR_0 over RCA_0.

Conjecture:[Clote '90][Simpson '99][Marcone]
FRA is equivalent to ATR_0 over RCA_0.



The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's lemma + RCA_0

ACA_0: Arithmetic Comprehension + RCA_0
$\Leftrightarrow$ "for every set X, X' exists".

ATR_0: Arithmetic Transfinite recursion + ACA_0.
$\Leftrightarrow$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".

Π^1_1-CA_0: Π^1_1-Comprehension + ACA_0.
$\Leftrightarrow$ "$\forall X$, the hyper-jump of X exists".

Fraïssé's conjecture.

Claim

RCA_0+FRA is the least system where it is possible to develop a reasonable theory of embeddability of linear orderings.

Fraïssé's conjecture.

Claim

RCA_0+FRA is the least system where it is possible to develop a reasonable theory of embeddability of linear orderings.

Theorem ([M. 05])

The following are equivalent over RCA_0

- *FRA;*
- *Every scattered lin. ord. is a finite sum of indecomposables;*
- *Every indecomposable lin. ord. is either an ω-sum or an ω^*-sum of indecomposable l.o. of smaller rank.*
- *Jullien's characterization of extendible linear orderings*

A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

Theorem $(*)$**:**[Laver '72]
For every countable $\mathcal{L}$, there exists $n_{\mathcal{L}} \in \mathbb{N}$, such that:
If $\mathcal{L}$ is colored with finitely many colors,
there is an embedding $\mathcal{L} \to \mathcal{L}$ whose image has at most $n_{\mathcal{L}}$ colors.

A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

Theorem $(*)$**:**[Laver '72]
For every countable $\mathcal{L}$, there exists $n_{\mathcal{L}} \in \mathbb{N}$, such that:
If $\mathcal{L}$ is colored with finitely many colors,
there is an embedding $\mathcal{L} \to \mathcal{L}$ whose image has at most $n_{\mathcal{L}}$ colors.

Theorem ([M. 2005])

FRA is implied by Theorem $(*)$ over RCA_0.

A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

Theorem $(*)$**:**[Laver '72]
For every countable $\mathcal{L}$, there exists $n_{\mathcal{L}} \in \mathbb{N}$, such that:
If $\mathcal{L}$ is colored with finitely many colors,
there is an embedding $\mathcal{L} \to \mathcal{L}$ whose image has at most $n_{\mathcal{L}}$ colors.

Theorem ([M. 2005])

FRA is implied by Theorem $(*)$ over RCA_0.

Theorem ([Kach–Marcone–M.–Weiermann 2011])

FRA is equivalent to Theorem $(*)$ over RCA_0.

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$,
quotiented by the bi-embeddability relation, and
ordered by the embeddability relation.

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$, quotiented by the bi-embeddability relation, and ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$, quotiented by the bi-embeddability relation, and ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$,
quotiented by the bi-embeddability relation, and
ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
3. (This was used to prove that every hypearithmetic linear ordering is bi-embeddable with a computable one in [M. 05])

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$,
quotiented by the bi-embeddability relation, and
ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
3. (This was used to prove that every hypearithmetic linear ordering is bi-embeddable with a computable one in [M. 05])
4. FRA is equivalent to "$\forall$ ordinal $\alpha < \omega_1$ $(\mathbb{L}_\alpha$ is WQO)."

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$,
quotiented by the bi-embeddability relation, and
ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
3. (This was used to prove that every hypearithmetic linear ordering is bi-embeddable with a computable one in [M. 05])
4. FRA is equivalent to "$\forall$ ordinal $\alpha < \omega_1$ ($\mathbb{L}_\alpha$ is WQO)."

Question: Given α, what is the length of $\mathbb{L}_\alpha$?

Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$,
quotiented by the bi-embeddability relation, and
ordered by the embeddability relation.

1. [Laver 71] For countable α, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
3. (This was used to prove that every hypearithmetic linear ordering is bi-embeddable with a computable one in [M. 05])
4. FRA is equivalent to "$\forall$ ordinal $\alpha < \omega_1$ ($\mathbb{L}_\alpha$ is WQO)."

Question: Given α, what is the length of $\mathbb{L}_\alpha$?
Given α, what is the rank of $\mathbb{L}_\alpha$ as a well-founded poset?

Finite Hausdorff rank

Theorem ([Marcone, M 08])

The length of $\mathbb{L}_\omega$ *is* $\epsilon_{\epsilon_{\epsilon_{\dots}}}$,

Finite Hausdorff rank

Theorem ([Marcone, M 08])

The length of $\mathbb{L}_\omega$ *is* $\epsilon_{\epsilon_{\epsilon_{\dots}}}$,

where $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ *is the first fixed point of the function* $\alpha \mapsto \epsilon_\alpha$,

where ϵ_α *is the* $(\alpha + 1)$*st fixed point for the function* $\beta \mapsto \omega^\beta$.

Finite Hausdorff rank

Theorem ([Marcone, M 08])

The length of $\mathbb{L}_\omega$ *is* $\epsilon_{\epsilon_{\epsilon_{\dots}}}$,

where $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ *is the first fixed point of the function* $\alpha \mapsto \epsilon_\alpha$,

where ϵ_α *is the* $(\alpha+1)$*st fixed point for the function* $\beta \mapsto \omega^\beta$.

Note: $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the proof-theoretic ordinal of ACA^+,
where ACA^+ is the system $\mathsf{RCA}_0 + \forall X (X^{(\omega)}$ *exists*).
(So $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the least ordinal that ACA^+ ca**n't** prove is well-ordered.)

Finite Hausdorff rank

Theorem ([Marcone, M 08])

The length of $\mathbb{L}_\omega$ is $\epsilon_{\epsilon_{\epsilon_{\dots}}}$,
where $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the first fixed point of the function $\alpha \mapsto \epsilon_\alpha$,
where ϵ_α is the $(\alpha+1)$st fixed point for the function $\beta \mapsto \omega^\beta$.

Note: $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the proof-theoretic ordinal of ACA^+,
where ACA^+ is the system $RCA_0 + \forall X (X^{(\omega)}$ *exists*).
(So $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the least ordinal that ACA^+ ca**n't** prove is well-ordered.)

Theorem ([Marcone, M 08])

That $\mathbb{L}_\omega$ is a WQO,

- *follows from ACA^+ + "$\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is well-ordered",*
- *but not from ACA^+.*

1. Background on WQOs

2. WQOs in Proof Theory
   - Kruskal's theorem and the graph-minor theorem
   - Linear orderings and Fraïssé's Conjecture

3. WPOs in Computability Theory

Recall: $o(\mathcal{P}) = \sup\{\mathrm{ordType}(P, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

complexity of maximal order types

Recall: $o(\mathcal{P}) = \sup\{\mathrm{ordType}(P, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

Theorem: [De Jongh, Parikh 77]
Every WPO $\mathcal{P}$ has a linearization of order type $o(\mathcal{P})$.

complexity of maximal order types

Recall: $o(\mathcal{P}) = \sup\{\mathrm{ordType}(P, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

Theorem: [De Jongh, Parikh 77]
Every WPO $\mathcal{P}$ has a linearization of order type $o(\mathcal{P})$.

We call such a linearization, a *maximal linearization* of $\mathcal{P}$.

Recall: $o(\mathcal{P}) = \sup\{\mathrm{ordType}(P, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

Theorem: [De Jongh, Parikh 77]
Every WPO $\mathcal{P}$ has a linearization of order type $o(\mathcal{P})$.

We call such a linearization, a *maximal linearization* of $\mathcal{P}$.

Such linearizations have been found by different methods in different examples.

complexity of maximal order types

Recall: $o(\mathcal{P}) = \sup\{\mathrm{ordType}(P, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{P}\}$.

Theorem: [De Jongh, Parikh 77]
Every WPO $\mathcal{P}$ has a linearization of order type $o(\mathcal{P})$.

We call such a linearization, a *maximal linearization* of $\mathcal{P}$.

Such linearizations have been found by different methods in different examples.

Question [Schmidt 1979]:
Is the length of a computable WPO computable?

Computable Length

Q: Is the length of a computable WPO, computable?

Computable Length

Q: Is the length of a computable WPO, computable?

We mentioned that $o(\mathcal{P}) + 1 = \text{rk}(\mathbb{B}\text{ad}(\mathcal{P}))$, where

$$\mathbb{B}\text{ad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j \ (x_i \not\leq_P x_j)\},$$

Since $\mathbb{B}\text{ad}(\mathcal{P})$ is computable and well-founded, it has rank $< \omega_1^{CK}$.
So, $o(\mathcal{P})$ is a computable ordinal.

Computable Length

Q: Is the length of a computable WPO, computable?

We mentioned that $o(\mathcal{P})+1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{P}))$, where

$$\mathbb{B}\mathrm{ad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j\ (x_i \not\leq_P x_j)\},$$

Since $\mathbb{B}\mathrm{ad}(\mathcal{P})$ is computable and well-founded, it has rank $< \omega_1^{CK}$.
So, $o(\mathcal{P})$ is a computable ordinal.

Q:
Does every computable WPO have a computable maximal linearization?

A computable maximal linearization

Theorem ([M 2007])

Every computable WPO has a computable maximal linearization.

A computable maximal linearization

Theorem ([M 2007])

Every computable WPO has a computable maximal linearization.

Q: Can we find them uniformly?

A computable maximal linearization

Theorem ([M 2007])

Every computable WPO has a computable maximal linearization.

Q: Can we find them uniformly?

Theorem ([M 2007])

There is computable procedure that

given $\mathcal{P}$ produces a linearization $\mathcal{L}$ such that for some δ

$$\omega^{\delta} \leqslant \mathcal{L} \leqslant o(\mathcal{P}) < \omega^{\delta+1}.$$

A computable maximal linearization

Theorem ([M 2007])

Every computable WPO has a computable maximal linearization.

Q: Can we find them uniformly?

Theorem ([M 2007])

There is computable procedure that

given $\mathcal{P}$ produces a linearization $\mathcal{L}$ such that for some δ

$$\omega^{\delta} \leqslant \mathcal{L} \leqslant o(\mathcal{P}) < \omega^{\delta+1}.$$

Theorem ([M 2007])

Let **a** *be a Turing degree. TFAE:*

1. **a** *uniformly computes maximal linearizations of computable WPOs.*
2. **a** *uniformly computes $0^{(\beta)}$ for every $\beta < \omega_1^{CK}$.*

The height of a WPO

We denote by Ch($\mathcal{P}$) the collection of all chains of $\mathcal{P}$.

The height of a WPO

We denote by Ch($\mathcal{P}$) the collection of all chains of $\mathcal{P}$.

$\mathcal{P}$ is a WPO $\Rightarrow$ all its chains are well-orders.

The height of a WPO

We denote by $\mathsf{Ch}(\mathcal{P})$ the collection of all chains of $\mathcal{P}$.

$\mathcal{P}$ is a WPO $\Rightarrow$ all its chains are well-orders.

Definition

If $\mathcal{P}$ is well founded, its *height* is

$$\mathsf{ht}(\mathcal{P}) = \sup\{\alpha : \exists\mathcal{C} \in \mathsf{Ch}(\mathcal{P})\, \alpha = \mathrm{ordType}(\mathbb{L})\}.$$

The height of a WPO

We denote by $\mathsf{Ch}(\mathcal{P})$ the collection of all chains of $\mathcal{P}$.

$\mathcal{P}$ is a WPO $\Rightarrow$ all its chains are well-orders.

Definition

If $\mathcal{P}$ is well founded, its *height* is

$$\mathsf{ht}(\mathcal{P}) = \sup\{\alpha : \exists \mathcal{C} \in \mathsf{Ch}(\mathcal{P})\, \alpha = \mathrm{ordType}(\mathbb{L})\}.$$

Theorem: [Wolk 1967]
If $\mathcal{P}$ is a WPO, there exists $\mathcal{C} \in \mathsf{Ch}(\mathcal{P})$ with order type $\mathsf{ht}(\mathcal{P})$.

The height of a WPO

We denote by $\mathsf{Ch}(\mathcal{P})$ the collection of all chains of $\mathcal{P}$.

$\mathcal{P}$ is a WPO $\Rightarrow$ all its chains are well-orders.

Definition

If $\mathcal{P}$ is well founded, its *height* is

$$\mathsf{ht}(\mathcal{P}) = \mathsf{sup}\{\alpha : \exists \mathcal{C} \in \mathsf{Ch}(\mathcal{P})\, \alpha = \mathrm{ordType}(\mathbb{L})\}.$$

Theorem: [Wolk 1967]
If $\mathcal{P}$ is a WPO, there exists $\mathcal{C} \in \mathsf{Ch}(\mathcal{P})$ with order type $\mathsf{ht}(\mathcal{P})$.

Such a chain is called a *maximal chain* of $\mathcal{P}$.

The height of a WPO

We denote by $\mathsf{Ch}(\mathcal{P})$ the collection of all chains of $\mathcal{P}$.

$\mathcal{P}$ is a WPO $\Rightarrow$ all its chains are well-orders.

Definition

If $\mathcal{P}$ is well founded, its *height* is

$$\mathsf{ht}(\mathcal{P}) = \sup\{\alpha : \exists \mathcal{C} \in \mathsf{Ch}(\mathcal{P})\, \alpha = \mathrm{ordType}(\mathbb{L})\}.$$

Theorem: [Wolk 1967]
If $\mathcal{P}$ is a WPO, there exists $\mathcal{C} \in \mathsf{Ch}(\mathcal{P})$ with order type $\mathsf{ht}(\mathcal{P})$.

Such a chain is called a *maximal chain* of $\mathcal{P}$.

Q: How difficult is it to compute maximal chains?

Computing maximal chains

Theorem ([Marcone-Shore 2010])

Every computable WPO $\mathcal{P}$ has a hyperarithmetic maximal chain.

(Recall: $X \subseteq \omega$ is hyperarithmetic iff it's Δ^1_1.)

Computing maximal chains

Theorem ([Marcone-Shore 2010])

Every computable WPO $\mathcal{P}$ has a hyperarithmetic maximal chain.

(Recall: $X \subseteq \omega$ is hyperarithmetic iff it's Δ^1_1.)

Maximal chains aren't easy to compute:

Computing maximal chains

Theorem ([Marcone-Shore 2010])

Every computable WPO $\mathcal{P}$ has a hyperarithmetic maximal chain.

(Recall: $X \subseteq \omega$ is hyperarithmetic iff it's Δ^1_1.)

Maximal chains aren't easy to compute:

Theorem ([Marcone–M.–Shore 2012])

Let $\alpha < \omega_1^{\mathrm{CK}}$.
There exists a computable WPO $\mathcal{P}$ such that

$$0^{(\alpha)} \text{ does not compute any maximal chain of } \mathcal{P}.$$

Computing maximal chains

Maximal chains are not easy to compute,

Computing maximal chains

Maximal chains are not easy to compute,

but almost everybody can compute them.

Computing maximal chains

Maximal chains are not easy to compute,

but almost everybody can compute them.

Theorem ([Marcone-M.-Shore 2012])

Let $G \in 2^\omega$ be hyperarithmetically generic.

Then G can compute a maximal chain in every computable WPO.

Computing maximal chains

Maximal chains are not easy to compute,

but almost everybody can compute them.

Theorem ([Marcone-M.-Shore 2012])

Let $G \in 2^\omega$ be hyperarithmetically generic.

Then G can compute a maximal chain in every computable WPO.

Pf:

- The key observation is that all downward closed subsets of P are computable.

Computing maximal chains

Maximal chains are not easy to compute,

but almost everybody can compute them.

Theorem ([Marcone-M.-Shore 2012])

Let $G \in 2^\omega$ be hyperarithmetically generic.

Then G can compute a maximal chain in every computable WPO.

Pf:

- The key observation is that all downward closed subsets of P are computable.
- Suppose that $\mathcal{P}$ has cofinality $\omega^{\alpha+1}$.
- Then, build an operator $\Phi_\alpha^{\mathcal{P},G}$, that returns a sequence of computable sub-partial orderings $P_0 \leqslant P_1 \leqslant \ldots$, such that, if G is generic, then infinitely many of the P_i will have cofinality ω^α.

Computing maximal chains

Maximal chains are not easy to compute,

but almost everybody can compute them.

Theorem ([Marcone-M.-Shore 2012])

Let $G \in 2^\omega$ be hyperarithmetically generic.

Then G can compute a maximal chain in every computable WPO.

Pf:

- The key observation is that all downward closed subsets of P are computable.
- Suppose that $\mathcal{P}$ has cofinality $\omega^{\alpha+1}$.
- Then, build an operator $\Phi_\alpha^{\mathcal{P},G}$, that returns a sequence of computable sub-partial orderings $P_0 \leqslant P_1 \leqslant ...$, such that, if G is generic, then infinitely many of the P_i will have cofinality ω^α.
- Then use effective transfinite recursion.