Saving Truth from Orthodoxy

Better Logic Through Algebra, Probability, and Dynamical Systems

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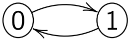
Diversity Is the Savior of Logical Truth

I. There is important MATHEMATICAL DIVERSITY in what we call 'logic' – 4 different classes of problems:
   - Arithmetic
   - Algebra (equations)
   - Dynamical systems
   - Probability

II. In arithmetic every formula has an elementary value (like 0 or 1). But the other systems return more complex objects (sets, graphs, polynomials):

$\{0, 1\}$ $\{\}$  $\theta_3 + \theta_1\theta_2 - \theta_1\theta_3$

Certain features of these objects seem paradoxical.

III. But when logic problems are properly classified, and unorthodox objects like these are accepted as truth values, many paradoxes disappear.*

*The TRUTH FAIRY replaces paradoxes with complex truth values.

Does the Truth Fairy Exist?

GEORGE BOOLE 1815–1864

Lies, Damn Lies, and Interpretations

- My Liar sentence: x: This sentence x is false.
- As a definition: $x :: \neg x,\ x \in \{\texttt{TRUE}, \texttt{FALSE}\}$
- Two interpretations: equation or recurrence relation
  - Many senses of $=$ (test, assignment, constraint)
  - In C program, `x==1-x` and `x=1-x` are different.
  - Mathematica `Solve[x==1-x]` different again.
- As an equation: $x = 1 - x,\ x \in \{0, 1\}$
  - From Boole: $\texttt{TRUE} \mapsto 1$; $\texttt{FALSE} \mapsto 0$; $\neg x \mapsto 1 - x$.
- As a recurrence: $x_{t+1} \Leftarrow 1 - x_t,\ x \in \{0, 1\}$
  - Assignment introduces an arrow of time.
- What kinds of truth values do we get from these two interpretations?

Solving Equations Gives a Set of Solutions

- A logical AXIOM is a polynomial EQUATION:
  - LOGICAL NOTATION: A judgment $\vdash \psi$ that asserts the truth of its content (ψ in propositional calculus)
  - POLYNOMIAL NOTATION: An equation $q = 0$ where q is $1 - \text{POLY}(\psi)$ using Boole's translation of logic:
    - $\text{POLY}(\neg p) = 1 - p$; $\text{POLY}(p \to q) = 1 - p + pq$; etc.
    - $\psi = \text{TRUE} \rightsquigarrow \text{POLY}(\psi) = 1 \rightsquigarrow 1 - \text{POLY}(\psi) = 0$
- The SOLUTION SET to polynomial equations gives the possible values of an objective formula ϕ subject to some axioms $\vdash \psi_1, \ldots, \vdash \psi_m$:

$$\mathcal{S}_Q(p) \equiv \{\, p(\mathbf{x}) : x_i \in \{0,1\},\ q_j(\mathbf{x}) = 0 \,\}$$

  with $\mathbf{x} = (x_1, \ldots, x_n)$; $p, q \in \mathbb{R}[\mathbf{x}]$; $p = \text{POLY}(\phi)$; each $q_j = 1 - \text{POLY}(\psi_j)$; $Q = \{q_1, \ldots, q_m\}$.
- This solution set must be a subset of the set of elementary values: for 2-valued logic $\mathcal{S}_Q(p) \subseteq \{0,1\}$.

Solution Sets Are Good Truth Values

- Solution set $\mathcal{S}_Q(\textsc{Poly}(\phi))$ gives truth value of the objective ϕ subject to the axioms $\vdash \psi_j$ given as Q:
  - $\{1\}$ ϕ is a THEOREM.
  - $\{0\}$ ϕ is the negation of a theorem.
  - $\{0,1\}$ ϕ is AMBIGUOUS
  - $\{\}$ ϕ is UNSATISFIABLE: the axioms $\vdash \psi_j$ are inconsistent
- Inverse sets of polynomials are useful:
  - $\mathcal{S}_Q^{-1}(\{1\})$: the set of all theorems entailed by axioms Q
  - $\mathcal{S}_Q^{-1}(\{0\})$: the *ideal* (algebraic geometry) Q generates
- This logic is PARACONSISTENT and PARACOMPLETE.
  - SOME INCLUDED MIDDLE: truth values come from the POWER SET $\{\{0\},\{1\},\{0,1\},\{\}\}$ of the set $\{0,1\}$.
    - Different idea from adding new elementary objects (like $\frac{1}{2}$ or 2), as usual in 'multivalued' logics.
  - NO EXPLOSION FROM CONTRADICTION: inconsistent axioms give every objective the empty solution set: so *nothing* is declared a theorem, not *everything*.

Sets of Solution Sets Give Modal Logic

$s \equiv \mathcal{S}_Q(\textsc{Poly}(\phi))$				Description of ϕ given axioms $\vdash \psi_j$ in Q		
$\{0\}$	$\{1\}$	$\{0,1\}$	$\{\}$	Set	Modal	Natural language
	•			$s = \{1\}$	$\Box(\phi)$	necessarily true
•		•	•	$s \neq \{1\}$	$\neg\Box(\phi)$	not necessarily true
•				$s = \{0\}$	$\Box(\neg\phi)$	necessarily false
	•	•	•	$s \neq \{0\}$	$\neg\Box(\neg\phi)$	not necessarily false
	•	•		$1 \in s$	$\Diamond(\phi)$	possibly true
•			•	$1 \notin s$	$\neg\Diamond(\phi)$	not possibly true
•		•		$0 \in s$	$\Diamond(\neg\phi)$	possibly false
	•		•	$0 \notin s$	$\neg\Diamond(\neg\phi)$	not possibly false
			•	$s = \{\}$	$\oslash(\phi)$	necessarily unsatisfiable
		•		$\lvert s\rvert > 1$	$\bowtie(\phi)$	necessarily ambiguous
•	•			$\lvert s\rvert = 1$	$\boxdot(\phi)$	determinate
•	•	•	•	$s \subseteq \{0,1\}$	$\heartsuit(\phi)$	$\{0,1\}$-compatible

- We reject $\Diamond(\phi) \equiv \neg\Box(\neg\phi)$: 'not necessarily false' allows inconsistent axioms, 'possibly true' does not.
- We reject $\Box(\phi) \equiv \phi$: $\Box(\phi)$ depends on axioms in Q but ϕ itself does not. Better: $\Box(\phi|Q)$, $\Diamond(\phi|Q)$, etc.

Curry's Dynamical System

x: If this sentence x is true then y is true.

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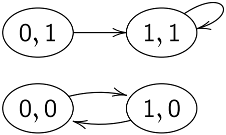
0. Definition: $x :: x \rightarrow y$ with $x, y \in \{\texttt{TRUE}, \texttt{FALSE}\}$
   - Boolean Poly $(x \rightarrow y) = 1 - x + xy$ with $x, y \in \{0, 1\}$
1. As recurrence: $x_{t+1} \Leftarrow 1 - x_t + x_t y_t$ with $x, y \in \{0, 1\}$
   - Dynamical system with state (x, y), transition graph:



   - There is one fixed point $(x, y) = (1, 1)$.
   - Because of the periodic cycle it seems PARADOXICAL when $y = 0$ (i.e. the consequent in $x \rightarrow y$ is false).
2. As equation: $x = 1 - x + xy$ with $x, y \in \{0, 1\}$
   - Solution sets $\mathcal{S}_Q(x) = \{1\}$ and $\mathcal{S}_Q(y) = \{1\}$

- Interpretation #1 adds value: shows oscillating cycle

Kripke's Watergate Dynamical System

x: That sentence y is false (J: "Most Nixon assertions false")

y: That sentence x is true (N: "Everything Jones says true")

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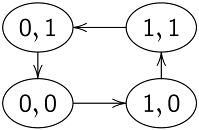
0. Definition: $x :: \neg y,\ y :: x$ with $x, y \in \{\text{TRUE}, \text{FALSE}\}$
   - Boolean Poly $(\neg y) = 1 - y$
1. As recurrences: $x_{t+1} \Leftarrow 1 - y_t,\ y_{t+1} \Leftarrow x_t$; $x, y \in \{0, 1\}$
   - Dynamical system with state (x, y), transition graph:



   - There are no fixed points.
   - Periodic cycle: every state seems PARADOXICAL.
2. As equations: $x = 1 - y$ and $y = x$ with $x, y \in \{0, 1\}$
   - Solution sets $\mathcal{S}_Q(x) = \{\}$ and $\mathcal{S}_Q(y) = \{\}$

- Interpretation #1 adds value: pattern of infeasibility

Gödel's Dynamical System

x: This formula x is true if and only if it is not provable.

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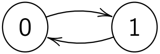
0. Definition: $x :: \neg\textsc{Provable}(x)$; $x \in \{\text{TRUE}, \text{FALSE}\}$
   - Solution set $\{1\}$ means 'provable'; here no axioms.
   - Revised definition $x :: \left(\mathcal{S}_{\{\}}(x) \neq \{1\}\right)$ or $x :: \neg\Box(x)$.
   - But $\mathcal{S}_{\{\}}(x)$ is just $\{x\}$. Then $(\{x\} \neq \{1\})$ is true when $x = 0$ and false when $x = 1$: its value is $1 - x$.
   - Re-revised definition $x :: 1 - x$ (as the Liar $x :: \neg x$).
1. As recurrence: $x_{t+1} \Leftarrow 1 - x_t$ with $x \in \{0, 1\}$
   - Dynamical system with state x and transition graph:



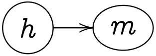
   - There are no fixed points.
   - Periodic cycle: Gödel called paradox UNDECIDABLE.
2. As equation: $x = 1 - x$ with $x \in \{0, 1\}$
   - Solution set $\mathcal{S}_Q(x) = \{\}$

The Logic of Parametric Probability

- My PARAMETRIC PROBABILITY ANALYSIS method solves many problems in logic, including:
  - Counterfactual conditionals
  - Probabilities of formulas in the propositional calculus
  - Aristotle's syllogisms
  - Smullyan's puzzles with liars and truth-tellers
- Solutions to probability queries are polynomials in the parameters $\theta_1, \theta_2, \ldots$ used to specify probabilities (distinct from the primary variables $x_1, \ldots, x_n$ whose probabilities are specified).
- These θ-polynomials can be used for secondary analysis such as linear and nonlinear optimization, search, and general algebra.
- Details are on arXiv.org. With some probability $0 \leqslant \theta \leqslant 1$, I will present at LC2012 in Manchester.

Probability: Counterfactual Conditionals

- Goodman's counterfactual piece of butter:
  B1. If it had been heated it would have melted.
  B2. If it had been heated it would not have melted.
- Undesired: $\text{Poly}\left((h \rightarrow m) \wedge (h \rightarrow \neg m)\right) = 1 - h$.
  B1, B2 as material implication say $\neg h$, not heated.
- Probability network ($h$eat, $m$elt; $0 \leqslant \theta_1, \theta_2, \theta_3 \leqslant 1$):



h	$\text{Pr}_0(h)$
1	θ_1
0	$1 - \theta_1$

$\text{Pr}_0(m \mid h)$		
h	$m = 1$	$m = 0$
1	θ_2	$1 - \theta_2$
0	θ_3	$1 - \theta_3$

h	m	$\text{Pr}(h, m)$
1	1	$\theta_1\theta_2$
1	0	$\theta_1 - \theta_1\theta_2$
0	1	$\theta_3 - \theta_1\theta_3$
0	0	$1 - \theta_1 - \theta_3 + \theta_1\theta_3$

  - $\text{Pr}(m = 1 \mid h = 1) \Rightarrow (\theta_1\theta_2) / (\theta_1)$ $\quad \ldots 0/0$ if $\theta_1 = 0$
  - $\text{Pr}(h = 1) \Rightarrow \theta_1, \quad \text{Pr}(m = 1) \Rightarrow \theta_3 + \theta_1\theta_2 - \theta_1\theta_3$
- B1 is constraint $\theta_2 = 1$; B2 is incompatible $\theta_2 = 0$.
- B1 and B2 constrain output $\text{Pr}(m \mid h)$, not $\text{Pr}(h)$.
- Results are quotients of polynomials in $\mathbb{R}[\theta_1, \theta_2, \theta_3]$.

Post-Paradox Paradigm for Logic

DENNIS RITCHIE 1941–2011 · KEN THOMPSON 1943–

Upgrade to Logicism 2.0

1.0: One Step

MATHEMATICS
Number theory
Arithmetic

↑

LOGIC
First-order logic
Set theory

2.0: Layered Software and Systems

HIGH-LEVEL (USER) LOGIC
First-order logic, theorem proving
Full set theory and number theory
Combined logic and probability

↑

INTERMEDIATE MATHEMATICS
General computer programming
Algebra, polynomial equations, finite sets
Dynamical systems, probability networks

↑

LOW-LEVEL (MACHINE) LOGIC
Finite-integer arithmetic, logic gates
Leibniz, Pascal calculators in 1600s
Programmable digital computers

Conclusion: Paradox Lost, Logic Found

- What we call 'logic' includes four different classes of mathematical problems: arithmetic, algebra, dynamical systems, and probability.
- When logic problems are appropriately classified and analyzed (as if by the Truth Fairy), things that once seemed paradoxical or undecidable become routine.
- In particular, some logic problems specify dynamical systems with periodic orbits. These results are not pathological and they do not render formal reasoning incomplete in any fundamental way.
- For sound logic, celebrate mathematical diversity:
  - *Say it loud: polynomial and proud!*
  - *Sets are solutions too*
  - *Probability ♡ Logic*
  - *Oscillation is not a crime*

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Extra Slides

GOTTFRIED WILHELM LEIBNIZ 1646–1716

Paradoxes Get in the Way of Applications

For MEDICAL DECISION MAKING I need formal reasoning systems that deliver a few important features:

- Reasoning under uncertainty and ambiguity
- Learning from observations and data
- Verifiable correctness
- Introspection and metalevel reasoning

But these features are exactly the subjects of several PARADOXES and other challenges in mathematical logic, decision theory, and probability theory.

- What seems paradoxical to logicians and why?
- Can we solve these issues?

Philosophy: From Pythagoras to Gödel

- Gödel took as his prototype the Liar sentence of Eubulides (ca. 350 B.C., with Aristotle).
- We can also learn from Pythagoras (ca. 500 B.C.).
  - $a^2 + b^2 = c^2$ gives irrational c for some integers a, b.
  - The Pythagoreans regarded only 'natural numbers' as acceptable; they drowned Hippasus at sea for $\sqrt{2}$!
  - It took time to accept $\sqrt{2}$, $\sqrt{-1}$, etc. as numbers; we still insult them as 'irrational' (αλόγος, not logical) and 'imaginary' (not 'real').
- Is Gödel's 'undecidable' like Pythagoras' 'irrational'? Is the orthodox view to accept only THEOREM or NEGATION-OF-THEOREM as answers too narrow?
  1. Are there some mathematical objects that make sense as truth values for self-referential formulas like Gödel's, Russell's, etc.? Yes!
  2. (But can we do interesting logic with them?)

Arithmetic: Polynomial Propositions

The PROPOSITIONAL CALCULUS can be viewed as arithmetic where logical operators act on polynomials.

- Logical formula $\phi(x_1, \ldots, x_n) \mapsto p \in \mathbb{R}[x_1, \ldots, x_n]$.
  - Function POLY maps formulas to polynomials.
- Boole showed how to interpret logical operators for polynomial arguments $(p,\ q)$ and polynomial values:

$$\begin{aligned} \text{TRUE} &\mapsto 1 \\ \text{FALSE} &\mapsto 0 \\ \neg p &\mapsto 1 - p \end{aligned} \qquad \begin{aligned} p \wedge q &\mapsto p \times q \\ p \vee q &\mapsto p + q - pq \\ p \rightarrow q &\mapsto 1 - p + pq \\ p \leftrightarrow q &\mapsto 1 - p - q + 2pq \end{aligned}$$

  - Each $x_i \in \{0, 1\}$, so $x_i = x_i^2$: can substitute x_i for x_i^2
- For example (using $\mathbb{R}[x, y]$ as elementary set):
  - $x \wedge (x \rightarrow y) \Rightarrow x \times (1 - x + xy) \Rightarrow x - x^2 + x^2 y \Rightarrow xy$
  - So $\text{POLY}\,(x \wedge (x \rightarrow y)) = xy$ just as $\text{POLY}\,(2 + 2) = 4$
  - Inverse: $\text{POLY}^{-1}\,(xy) = \{x \wedge y, x \wedge (x \rightarrow y), \ldots\}$

Using Inverse Evaluation Functions

- Inverse arithmetical evaluation gives the LOGICAL PREIMAGE of a polynomial:

$$\textsc{Poly}^{-1}(p) \equiv \{\phi : \phi \in \mathcal{L}_a,\ \textsc{Poly}(\phi) = p\}$$

These logical formulas share the same truth table.
  - E.g. $\textsc{Poly}^{-1}(xy) = \{x \wedge y, x \wedge (x \rightarrow y), \ldots\}$
- Inverse algebraic evaluation gives the set of all polynomials with a common solution set given A:

$$\mathcal{S}_Q^{-1}(s) \equiv \{p : p \in K[x_1, \ldots, x_n],\ \mathcal{S}_Q(p) = s\}$$

Therefore $\mathcal{S}_Q^{-1}(\{1\})$ is the SET OF ALL THEOREMS entailed by the axioms in Q (in polynomial form).
  - Like the *ideal* $\mathcal{S}_Q^{-1}(\{0\})$ this set has a closed form.
  - Using $\mathbb{F}_2$, the set $\mathcal{S}_Q^{-1}(\{1\}) \subset \mathbb{F}_2[x_1, \ldots, x_n]$ is finite.
  - The logical preimage $\textsc{Poly}^{-1}(p)$ gives logical notation for each polynomial theorem $p \in \mathcal{S}_Q^{-1}(\{1\})$.

The Dynamic Topology of Truth

- In DYNAMICAL SYSTEMS the value of a formula is a state-transition graph. Each state is usually an elementary object or a vector or set of them.
- The topology of each graph specifies a truth value.
  - How many fixed points?
    - 0 INCONSISTENT
    - 1 CONSISTENT
    - $\geqslant 2$ CONTINGENT
  - Any nonconvergent orbits (periodic or infinite)?
    - **yes** UNSTEADY (These really bother logicians!)
    - **no** STEADY
  - Thus 6 categories of dynamic truth: meta-modalities that concern *stability* rather than *necessity*.
  - In each state, every formula has a usual solution set.
- A dynamical system can be solved for its fixed points (thus interpreted as a set of simultaneous equations).

The Logic of Parametric Probability

Two ways to apply PARAMETRIC PROBABILITY ANALYSIS:

- EMBEDDING: Probability tables copy truth tables.
  - E.g. for $\phi = A \to B$ add C and derived $\Pr(C \mid A, B)$:

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

$\rightsquigarrow$

A	B	$\Pr(C = \mathsf{T})$	$\Pr(C = \mathsf{F})$
T	T	1	0
T	F	0	1
F	T	1	0
F	F	1	0

  Ask $\Pr([A \to B])$, $\Pr(B \mid A)$, $\Pr(A \mid [A \to B])$, etc.
- DIRECT ENCODING: Conditional probabilities encode if/then statements (without material implication).
  - By clever factoring we can constrain $\Pr(B \mid A)$ without affecting $\Pr(A)$, and get the desired semantics for counterfactual conditionals.
- Solutions: polynomials in the parameters θ_i used to specify probabilities (with rational coefficients).
- Secondary analysis: optimization, search, etc.

Embedding: A Challenge in the Cards

Embedding allows reasoning about the probabilities of statements in the propositional calculus.

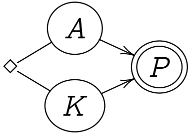
- A problem from Johnson-Laird told by Bringsjord:
  0. If one of the following is true then so is the other:
  1. There is a king in the hand iff there is an ace.
  2. There is a king in the hand.
  - Which is more likely, if either: the king or the ace?
- Logical formula for Sentence 0: $(K \leftrightarrow A) \leftrightarrow K$
- Query: Relative values of $\Pr(A = \mathsf{T})$ and $\Pr(K = \mathsf{T})$

Detour: easy resolution of illusion

- Johnson-Laird's 'illusory inference' problems are mostly about simplifying nested biconditionals.
- Boolean interpretation $\textsc{Poly}((K \leftrightarrow A) \leftrightarrow K) = A$.
- The ace is present with certainty if Sentence 0 holds; hence it is as likely or more likely than the king.

Probability Network, Embedded Logic

We can also use parametric probability with embedded propositional calculus to solve this ace-king problem.

- Binary variables A and K; add P for $(K \leftrightarrow A) \leftrightarrow K$
- Network graph: 

- Real parameters $0 \leqslant x_i \leqslant 1$ with $x_1 + x_2 + x_3 + x_4 = 1$.
- Component probabilities: $\Pr(A, K)$ is uninformative, $\Pr(P \mid A, K)$ copies truth table for $(K \leftrightarrow A) \leftrightarrow K$.

A	K	$\Pr_0(A, K)$
T	T	x_1
T	F	x_2
F	T	x_3
F	F	x_4

$\Pr_0(P \mid A, K)$			
A	K	$P = \mathrm{T}$	$P = \mathrm{F}$
T	T	1	0
T	F	1	0
F	T	0	1
F	F	0	1

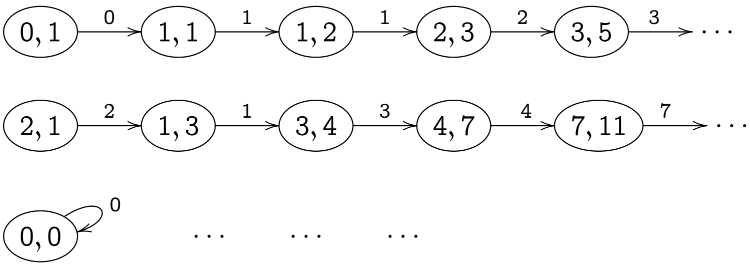
Primary and Secondary Analysis

We compare the probabilities of A versus K, given the condition P for the problem's assertion $(K \leftrightarrow A) \leftrightarrow K$.

- Primary analysis is symbolic probability inference:
  - $\Pr(A = \mathsf{T}) \Rrightarrow x_1 + x_2$
  - $\Pr(K = \mathsf{T}) \Rrightarrow x_1 + x_3$
  - $\Pr(P = \mathsf{T}) \Rrightarrow x_1 + x_2$
- Here, secondary analysis is linear optimization:
  - The difference $\Pr(A = \mathsf{T}) - \Pr(K = \mathsf{T})$ is $x_2 - x_3$.
  - We desire minimum and maximum values of $x_2 - x_3$ subject to $0 \leqslant x_i \leqslant 1$, $x_1 + x_2 + x_3 + x_4 = 1$, and the constraint $\Pr(P = \mathsf{T}) = 1$, hence $x_1 + x_2 = 1$.
  - By linear programming: minimum 0, maximum 1.
  - These bounds $0 \leqslant \Pr(A = \mathsf{T}) - \Pr(K = \mathsf{T}) \leqslant 1$ imply $\Pr(A = \mathsf{T}) \geqslant \Pr(K = \mathsf{T})$: the ace is at least as likely as the king (when $(K \leftrightarrow A) \leftrightarrow K$ holds).
- Many problems about the probabilities of logical formulas are also linear optimization problems.

The Familiar Fibonacci Numbers

Annotated state-transition graph using evolution function $F(x, y) : (y, x + y)$ and objective $G(x, y) : x$ extracted from the Fibonacci recurrence $x_{t+2} \Leftarrow x_t + x_{t+1}$



- Each ORBIT gives an infinite sequence of objective values. From $(0, 1)$ the usual $(0, 1, 1, 2, 3, 5, 8, \ldots)$.
- A unique FIXED POINT at $(0, 0)$ since $(0, 0) = F(0, 0)$
- All other orbits do not converge

Self-Referential Quadratic Equations

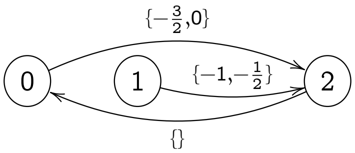
c is the number of real solutions to $2x^2 + 3x + c = 0$.

b is the number of real solutions to $y^2 + 6by + 11 = 0$.

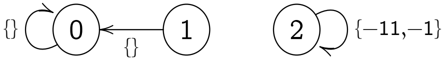
- As recurrences for b and c, state space $\{0, 1, 2\} \subset \mathbb{R}$:

$$c_{t+1} \Leftarrow \left|\{x : x \in \mathbb{R},\ c_t \in \mathbb{R},\ 2x^2 + 3x + c_t = 0\}\right|$$
$$b_{t+1} \Leftarrow \left|\{y : y \in \mathbb{R},\ b_t \in \mathbb{R},\ y^2 + 6b_t y + 11 = 0\}\right|$$

- Dynamical system for c (edges show solutions for x):

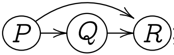


- Dynamical system for b (edges show solutions for y):



Outline: Diverse Systems and Solutions

- ARITHMETIC: $2+2 \Rightarrow 4$
- ALGEBRA (EQUATIONS):
  - Data: $x \in \mathbb{R}, x^2 = x$
  - Query: $\{x : x \in \mathbb{R}, x^2 = x\} \Rightarrow \{0, 1\}$
- DYNAMICAL SYSTEMS:
  - Data: $x \in \{0, 1\}$, $x_{t+1} \Leftarrow 1 - x_t$
  - Query: [Phase portrait of x] $\Rightarrow$ 

  - Query: [Orbit of x from $x_0 = 0$] $\Rightarrow (0, 1, 0, 1, \ldots)$
- PROBABILITY:
  - Data: $P, Q, R \in \{0, 1\}$; 

; $x, y, z \in \mathbb{R}$;

P	$\Pr_0(P)$
1	x
0	$1-x$

$\Pr_0(Q \mid P)$		
P	$Q=1$	$Q=0$
1	y	$1-y$
0	z	$1-z$

$\Pr_0(R \mid P, Q)$			
P	Q	$R=1$	$R=0$
1	1	1	0
1	0	0	1
0	1	1	0
0	0	1	0

  - Query: $\Pr(R=1) - \Pr(Q=1 \mid P=1) \Rightarrow$
    $1 - x - y + xy$ with $0 < x \leqslant 1$; $0 \leqslant y \leqslant 1$