Model Companion of Unstable Theories with an Automorphism

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Basic Set-up

Let $\mathcal{L}$ be a first-order language, and T be an $\mathcal{L}$-theory.

Let σ be a "new" unary function symbol, and let $\mathcal{L}_\sigma := \mathcal{L} \cup \{\sigma\}$.

Let $T_\sigma := T \cup \{$"σ is an $\mathcal{L}$-automorphism"$\}$.

Question: Does T_σ have a model companion in $\mathcal{L}_\sigma$?
(If it does, we denote the model companion by T_A.)

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History

Theorem (Kikyo, 2000)

If T is unstable without IP, then T_A does not exist.

Theorem (Kikyo-Shelah, 2002)

If T has SOP, then T_A does not exist.

Theorem (Kudaibergenov, ????)

If T is stable and has the fcp, then T_A does not exist.

Theorem (Baldwin-Shelah, 2003)

If T is stable, then T_A exists iff T does not admit obstructions.

Open Problem: What happens if T has IP?

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Proof Sketch.

Let $(M, \sigma) \models T_\sigma$ and $\langle a_i : i < \omega \rangle$ in M satisfy $a_i < a_{i+1} = \sigma(a_i)$.

Assuming T_A exists, extend (M, σ) to a sufficiently saturated model (N, σ) of T_A.

Let $p(x) := \{x > a_i : i < \omega\}$ and $\psi(x) := \exists y(a_0 < \sigma(y) < y < x)$.

In (N, σ),

1. $p(x) \vdash \psi(x)$
2. if $q(x)$ is a finite subset of $p(x)$, then $q(x) \nvdash \psi(x)$.

This is a contradiction to the saturation of (N, σ). □

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Linear Order

Definition

Let L be a linear order in the language $\mathcal{L}_O := \{<\}$. An $\mathcal{L}_O$-automorphism σ of L is called increasing if $\forall x(x < \sigma(x))$.

Definition

Let LO_σ^+ (DLO_σ^+) denote the $\mathcal{L}_{O,\sigma}$-theory of (dense) linear orders together with the axioms denoting "σ is an increasing $\mathcal{L}_O$-automorphism".

Theorem (P.)

LO_σ^+ has a model companion (namely DLO_σ^+) in $\mathcal{L}_{O,\sigma}$. Moreover, DLO_σ^+ eliminates quantifiers and is o-minimal.

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Ordered Abelian Groups

Definition

Let G be an ordered abelian group in the language $\mathcal{L}_{OG} := \{+, -, 0, <\}$. An $\mathcal{L}_{OG}$-automorphism σ of G is called (positive) increasing if $\forall x > 0 (x < \sigma(x))$.

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Let $ODAG^+_\sigma$ denote the $\mathcal{L}_{OG,\sigma}$-theory of ordered divisible abelian groups together with a (positive) increasing automorphism.

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Ordered Abelian Groups

Theorem (P.-Laskowski)

$ODAG_\sigma^+$ does not have a model companion in $\mathcal{L}_{OG,\sigma}$.

Proof Sketch.

Consider $(\mathbb{Q}, \sigma)$ where $\sigma(x) = 3x$. Clearly $(\mathbb{Q}, \sigma) \models \mathrm{ODAG}_\sigma^+$.
Let $\langle a_i : i < \omega \rangle$ in M satisfy $\sigma(a_i) = a_{i+1} = 3a_i$.

Extend $(\mathbb{Q}, \sigma)$ to (sufficiently saturated) $(N, \sigma) \models (\mathrm{ODAG}_\sigma^+)_A$.
Define

$$p(x) := \{x > a_i : i < \omega\} \text{ and } \psi(x) := \exists y(a_0 < y < x \wedge \sigma(y) = 2y).$$

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Multiplicative Ordered Abelian Groups

Recall that G is an ordered difference abelian group. So to get a model companion, we at least need to answer if following type of equations has a solution:

$$L(x) := (m_0 + m_1\sigma + \cdots + m_{k-1}\sigma^{k-1} + m_k\sigma^k)(x) = 0,$$

where $k \in \mathbb{N}, m_0, \ldots, m_k \in \mathbb{Z}$. Thus $L \in \mathbb{Z}[\sigma]$.
Such equations are called linear difference equations.

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MODAG and div-MODAG

Axiom OM: for each $L \in \mathbb{Z}[\sigma]$,

$$\Big(\forall x > 0\ (L(x) > 0)\Big) \bigvee \Big(\forall x > 0\ (L(x) = 0)\Big) \bigvee \Big(\forall x > 0\ (L(x) < 0)\Big).$$

Definition (P.)

An ordered difference abelian group is called multiplicative (denoted MODAG) if it satisfies Axiom OM.

Definition

A non-trivial MODAG is called divisible (denoted div-MODAG) if

$$\Big(\forall x (L(x) = 0)\Big) \vee \Big(\forall y \exists x (L(x) = y)\Big), \qquad \text{for } L \in \mathbb{Z}[\sigma].$$

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MODAG has a model companion (namely, div-MODAG) in $\mathcal{L}_{OG,\sigma}$. Moreover, div-MODAG eliminates quantifiers and is o-minimal.

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Direct sum of ordered difference abelian groups

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Let $\mathbf{G} = (G, +_G, -_G, 0_G, <_G, \sigma_G)$ and $\mathbf{H} = (H, +_H, -_H, 0_H, <_H, \sigma_H)$ be two ordered abelian groups with automorphism. Define a new ordered difference abelian group $\mathbf{G} \oplus \mathbf{H} = (G \oplus H, +, -, 0, <, \sigma)$ as follows:

- $g_1 \oplus h_1 + g_2 \oplus h_2 := (g_1 +_G g_2) \oplus (h_1 +_H h_2)$
- $0 := 0_G \oplus 0_H$
- $g_1 \oplus h_1 < g_2 \oplus h_2$
  $\iff$ either $(h_1 < h_2)$ or $(h_1 = h_2$ and $g_1 < g_2)$
- $\sigma(g \oplus h) := \sigma_G(g) \oplus \sigma_H(h)$

Clearly there are isomorphic copies of $\mathbf{G}$ and $\mathbf{H}$ inside $\mathbf{G} \oplus \mathbf{H}$, namely $\{g \oplus 0_H \mid g \in G\}$ and $\{0_G \oplus h \mid h \in H\}$.

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Direct sum of ordered difference abelian groups

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Let $\mathbf{G} = (G, +_G, -_G, 0_G, <_G, \sigma_G)$ and $\mathbf{H} = (H, +_H, -_H, 0_H, <_H, \sigma_H)$ be two ordered abelian groups with automorphism. Define a new ordered difference abelian group $\mathbf{G} \oplus \mathbf{H} = (G \oplus H, +, -, 0, <, \sigma)$ as follows:

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More model complete ordered difference abelian groups

Theorem (P.-Laskowski)

Let **G** *and* **H** *be models of model complete theories* T_G *and* T_H *of ordered abelian groups with (certain restricted class of) automorphism. Further assume that there are quantifier-free* $\mathcal{L}_{OG,\sigma}$*-formulas* $\theta_G(x)$ *and* $\theta_H(x)$ *that define* G *and* H *inside* $\mathbf{G} \oplus \mathbf{H}$*. Then the theory* $T_{G\oplus H}$ *of* $\mathbf{G} \oplus \mathbf{H}$ *is also model complete. In addition, if* T_G *and* T_H *eliminate quantifiers, then* $T_{G\oplus H}$ *also eliminates quantifiers.*

Example

The theory of the ordered abelian group $\mathbb{Q} \oplus \mathbb{Q}$, with automorphism σ defined as $\sigma(a \oplus b) = 2a \oplus 3b$, eliminates quantifiers, and is hence model complete.

More generally, any two distinct multiplicative automorphism in each coordinate will work!

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Ordered Fields

Definition

Let F be an ordered field in the language $\mathcal{L}_{OR} := \{+, -, \times, 0, 1, <\}$. An $\mathcal{L}_{OR}$-automorphism σ is said to be (eventually) increasing if $\exists y \forall x (x > y \implies x < \sigma(x))$.

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Let RCF_σ^+ denote the $\mathcal{L}_{OR,\sigma}$-theory of real-closed fields together with an (eventually) increasing automorphism.

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RCF_σ^+ does not have a model companion in $\mathcal{L}_{OR,\sigma}$.

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