Dominating and unbounded reals in Hechler extensions

Justin Palumbo

UCLA

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If V is a model of set theory and $V[G]$ is a generic extension, a real $d \in V[G] \cap \omega^\omega$ is called *dominating* if for every $f \in V \cap \omega^\omega$ we have $f \leq^* d$.

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A real $x \in V[G] \cap \omega^\omega$ is called *unbounded* if for every $f \in V \cap \omega^\omega$ we have $x \not\leq^* f$.

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Theorem (Baumgartner, Dordal, 1985)

Say $V \models \mathrm{CH}$. Let G be generic for the finite support iteration of $\mathbb{D}_{\mathrm{nd}}$. Then $V[G] \models \mathfrak{s} = \omega_1 \wedge \mathfrak{b} = 2^\omega$. In particular $\mathfrak{s} < \mathfrak{b}$ is consistent.

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They also introduced a rank analysis for $\mathbb{D}$ and showed that their theorem holds for the usual Hechler extension. It was an open question whether $\mathbb{D}$ and $\mathbb{D}_{\mathrm{nd}}$ are equivalent as forcing notions.

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Thus $\mathbb{D}$ and $\mathbb{D}_{\text{tree}}$ provide a counterexample to the natural Cantor-Bernstein theorem in the category of forcing notions.

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Let G be $\mathbb{D}_{\mathrm{tree}}$-generic over V. Let x be an unbounded real in $V[G]$. Then there is a dominating real $y \in V[G]$ so that $(\exists^\infty n) y(n) < x(n)$. (That is, $x \not\leq^* y$).

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Brendle and Löwe proved a dichotomy theorem for the possible reals in the extension by $\mathbb{D}_{\mathrm{tree}}$:

Theorem (Brendle and Löwe, 2009)

Every real added by $\mathbb{D}_{\mathrm{tree}}$ is either dominating or infinitely equal to some ground model real.

Motivated by this, they made an analogous dichotomy-style conjecture on the possible subforcings of $\mathbb{D}_{\mathrm{tree}}$:

Conjecture (Brendle and Löwe)

The only nontrivial subforcings of $\mathbb{D}_{\mathrm{tree}}$ are Cohen forcing $\mathbb{C}$ and $\mathbb{D}_{\mathrm{tree}}$ itself.

We can see now that this conjecture is false. Forcing with $\mathbb{D}_{\mathrm{tree}}$ adds a $\mathbb{D}$-generic real, which is neither equivalent to $\mathbb{D}_{\mathrm{tree}}$ nor to $\mathbb{C}$.

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Theorem (P.)

Let d be a $\mathbb{D}_{\mathrm{nd}}$-generic real, and suppose $y \in V[d]$ is dominating. Then there are $z_0, z_1 \in V \cap \omega^{\nearrow\omega}$ so that $z_0 \circ d \circ z_1 \leq^* y$.

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We can view this theorem as saying that d generates all the dominating reals in $V[d]$.

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Corollary

The structures $(V \cap \omega^\omega, \leq^*)$ *and* $(\mathcal{D}, {}^*\geq)$ *are cofinally isomorphic.*

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Using this fact, one can extend work of Laflamme (*"Bounding and dominating numbers of families of functions on* $\mathbb{N}$", 1993), and give new consistently achievable values of the following three cardinal characteristics for bounded $\mathcal{F} \subseteq \omega^\omega$:

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Here $\mathcal{F}^\downarrow \subseteq \omega^\omega$ is the set of functions dominating $\mathcal{F}$. (So if $\mathcal{F} = V \cap \omega^\omega$ then $\mathcal{F}^\downarrow = \mathcal{D}$.)

http://www.math.ucla.edu/~justinpa/

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Justin Palumbo, *"Unbounded and dominating reals in Hechler extensions."*
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Thank you.