Fragments of Martin's Maximum and weak square

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1. Introduction

1.1 weak square

Def. (Schimmerling)

For an unctble. card. λ and a card. $\mu \leq \lambda$,

$\Box_{\lambda,\mu} \equiv$ There exists $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ s.t.

- $\mathcal{C}_\alpha$ is a family of club subsets of α of o.t. $\leq \lambda$,
- $1 \leq |\mathcal{C}_\alpha| \leq \mu$,
- $c \in \mathcal{C}_\alpha$ & $\beta \in \mathrm{Lim}(c) \;\Rightarrow\; c \cap \beta \in \mathcal{C}_\beta$.

- $\Box_{\lambda,1} \;\Leftrightarrow\; \Box_\lambda$.
- $\Box_{\lambda,\lambda} \;\Leftrightarrow\; \Box^*_\lambda \;\Leftrightarrow\;$ "There is a special λ^+-Aronszajn tree."
- $\lambda^{<\lambda} = \lambda \;\Rightarrow\; \Box_{\lambda,\lambda}$.

1.2 forcing axioms and weak square

Fact (Cummings-Magidor)
Assume MM. Then we have the following:

(1) $\square_{\omega_1,\omega_1}$ fails.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

(3) If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,\mu}$ fails for all $\mu < \lambda$.

(4) If $\mathrm{cof}(\lambda) > \omega_1$, then $\square_{\lambda,\mu}$ fails for all $\mu < \mathrm{cof}(\lambda)$.

Fact (Cummings-Magidor)
"MM + (1) + (2)" is consistent:

(1) $\square_{\lambda,\lambda}$ holds for all λ with $\mathrm{cof}(\lambda) = \omega_1 < \lambda$.

(2) $\square_{\lambda,\mathrm{cof}(\lambda)}$ holds for all λ with $\mathrm{cof}(\lambda) > \omega_1$.

Fact (Todorčević, Magidor)
PFA implies the failure of $\square_{\lambda,\omega_1}$ for any λ.

Fact (Magidor)
PFA is consistent with that $\square_{\lambda,\omega_2}$ holds for all λ.

1.3 consequences of MM

$$\begin{array}{l} \text{MM} \Rightarrow \text{WRP} \Rightarrow (\dagger) \Rightarrow \text{Chang's Conjecture} \\ \ \Downarrow \\ \text{PFA} \end{array}$$

- WRP $\equiv$ For any $\lambda \geq \omega_2$ and any stationary $X \subseteq [\lambda]^\omega$
  there is $R \subseteq \lambda$ s.t.

  $$|R| = \omega_1 \subseteq R \quad \& \quad X \cap [R]^\omega \text{ is stationary.}$$

- $(\dagger) \equiv$ Every ω_1-stationary preserving poset is semi-proper.

- Chang's Conjecture
  $\equiv$ For any structure $\mathcal{M} = \langle \omega_2; \ldots \rangle$ there is $M \prec \mathcal{M}$ s.t.

  $$|M| = \omega_1 \quad \& \quad |M \cap \omega_1| = \omega.$$

We discuss how weak square is denied by (†) and Chang's Conjecture.

2. (†) and weak square

2.1 Rado's Conjecture

- Rado's Conjecture
  $\equiv$ Every non-special tree has a non-special subtree of size ω_1.

Fact

Rado's Conjecture implies (†).

Fact(Todorčević)

Rado's Conjecture is inconsistent with MM.

$$\begin{array}{ccc} & & \text{Rado's Conjecture} \\ & & \Downarrow \\ \text{MM} & \Longrightarrow & (\dagger) \end{array}$$

Fact (Todorčević, Todorčević-Torres)

Assume Rado's Conjecture. Then we have the following:

(1) $\square_{\omega_1,\omega}$ fails. If CH fails in addition, then $\square_{\omega_1,\omega_1}$ fails.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

(3) If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,\omega}$ fails.

(4) If $\mathrm{cof}(\lambda) > \omega_1$, then $\square_{\lambda,\mu}$ fails for all $\mu < \mathrm{cof}(\lambda)$.

Fact

"Rado's Conjecture + (1) + (2)" is consistent:

(1) $\square_{\lambda,\lambda}$ holds for all λ with $\mathrm{cof}(\lambda) = \omega_1 < \lambda$.

(2) $\square_{\lambda,\mathrm{cof}(\lambda)}$ holds for all λ with $\mathrm{cof}(\lambda) > \omega_1$.

The situation is almost similar as MM.

But the above facts are not sharp for λ with $\mathrm{cof}(\lambda) = \omega_1 < \lambda$.

2.2 result

Thm. (Veličković-S., S.)

Assume (†). Then we have the following:

(1) $\square_{\omega_1,\omega}$ fails. If CH fails in addition, then $\square_{\omega_1,\omega_1}$ fails.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

(3) If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,\omega}$ fails.
If λ is strong limit in addition, then $\square_{\lambda,\mu}$ fails for all $\mu < \lambda$.

(4) If $\mathrm{cof}(\lambda) > \omega_1$, then $\square_{\lambda,\mu}$ fails for all $\mu < \mathrm{cof}(\lambda)$.

Fact

"(†) + (1) + (2)" is consistent:

(1) $\square_{\lambda,\lambda}$ holds for all λ with $\mathrm{cof}(\lambda) = \omega_1 < \lambda$.

(2) $\square_{\lambda,\mathrm{cof}(\lambda)}$ holds for all λ with $\mathrm{cof}(\lambda) > \omega_1$.

Conjecture

Assume (†). If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,\mu}$ fails for all $\mu < \lambda$.

3. Chang's Conjecture and weak square

3.1 known fact and result

Fact (Todorčvić)
Chang's Conjecture implies the failure of $\square_{\omega_1}$.

Thm. (S.)
Chang's Conjecture is consistent with $\square_{\omega_1,2}$.

3.2 Outline of Proof of Thm.

Let κ be a measurable cardinal. We prove

$$\Vdash_{\mathrm{Col}(\omega_1,<\kappa)*\dot{\mathbb{P}}} \text{“Chang's Conjecture} + \square_{\omega_1,2}\text{”},$$

where $\mathbb{P}$ is the poset adding a $\square_{\omega_1,2}$-seq. by initial segments:

- $\mathbb{P}$ consists of all $p = \langle \mathcal{C}_\alpha \mid \alpha \leq \delta \rangle$ $(\delta < \omega_2)$ which is an initial segment of a $\square_{\omega_1,2}$-seq.
- $p \leq q$ iff $p \supseteq q$.

($\mathbb{P}$ is $<\omega_2$-Baire and forces $\square_{\omega_1,2}$.)

We must prove $\mathrm{Col}(\omega_1, <\kappa) * \dot{\mathbb{P}}$ forces Chang's Conjecture.

In $V^{\mathsf{Col}(\omega_1,<\kappa)}$ suppose

$p \in \mathbb{P}$,
$\dot{\mathcal{M}}$ is a $\mathbb{P}$-name for a structure on ω_2,
$\mathcal{N} := \langle \mathcal{H}_\theta, \in, p, \dot{\mathcal{M}} \rangle$.

It suffices to prove that in $V^{\mathsf{Col}(\omega_1,<\kappa)}$ there is $p^* \leq p$ and $N^* \prec \mathcal{N}$ s.t

- p^* is N^*-generic,
- $|N^* \cap \omega_2| = \omega_1$ & $|N^* \cap \omega_1| = \omega$.

(p^* forces that $N^* \cap \omega_2$ witnesses Chang's Conjecture for $\dot{\mathcal{M}}$.)

We construct a $\subseteq$-increasing seq. $\langle N_\xi \mid \xi < \omega_1 \rangle$ of ctble. elem. submodels of $\mathcal{N}$ and a descending seq. $\langle p_\xi \mid \xi < \omega_1 \rangle$ in $\mathbb{P}$ below p s.t.

- $N_0 \cap \omega_1 = N_1 \cap \omega_1 = \cdots = N_\xi \cap \omega_1 = \cdots$,
- p_ξ is N_ξ-generic, and $p_\xi \in N_{\xi+1}$,
- $\{p_\xi \mid \xi < \omega_1\}$ has a lower bound,

using some modification of the Strong Chang's Conjecture.

Then $N^* := \bigcup_{\xi<\omega_1} N_\xi$ and a lower bound p^* of $\{p_\xi \mid \xi < \omega_1\}$ are as desired.

Modification of the Strong Chang's Conjecture:

<u>**Lem.**</u> (In $V^{\mathrm{Col}(\omega_1,<\kappa)}$)
If $N \prec \mathcal{N}$ is ctble. and $\langle q_n \mid n < \omega \rangle$ is an $(N, \mathbb{P})$-generic seq., then

$\forall c \subseteq \sup(N \cap \omega_2)$: club, threads $\bigcup_{n<\omega} q_n$

$\exists d \subseteq \sup(N \cap \omega_2)$: club, threads $\bigcup_{n<\omega} q_n$

$\exists q^* \leq \bigcup_{n<\omega} q_n {}^\frown \langle \{c, d\} \rangle$ s.t.

$\mathrm{sk}^{\mathcal{N}}(N \cup \{p'\}) \cap \omega_1 = N \cap \omega_1$.

3.3 Question

We used a measurable cardinal to construct a model of Chang's Conjecture and $\square_{\omega_1,2}$. On the other hand, recall:

Fact (Silver, Donder)
Con (ZFC + Chang's Conjecture)
$\Leftrightarrow$ Con (ZFC + $\exists\omega_1$-Erdös cardinal).

Question
What is the consistency strength of
"Chang's Conjecture + $\square_{\omega_1,2}$" ?