Reuniting the Antipodes:

Bringing together Nonstandard and Constructive Analysis

Sam Sanders[1]

Madison, WI, April 2, 2012

[1]This research is generously supported by the John Templeton Foundation.

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We present a notion of constructive computability directly based on Nonstandard Analysis.

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3. Levels of infinity (Stratified NSA).

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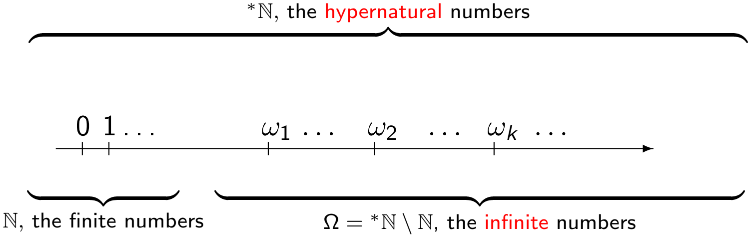

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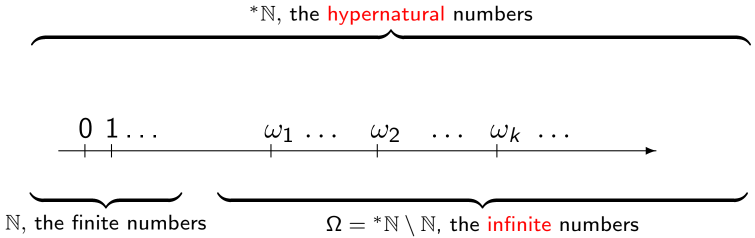

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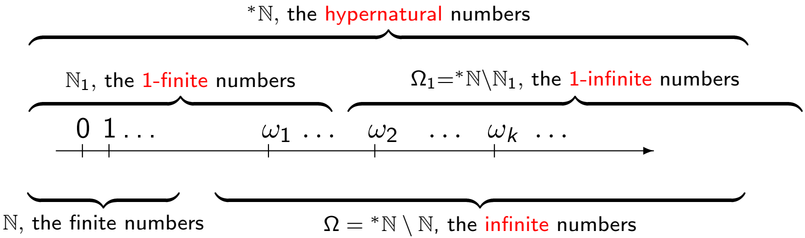

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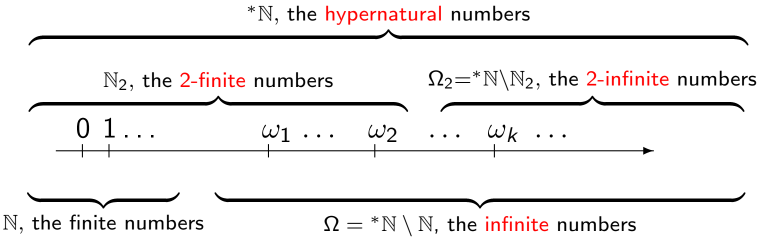

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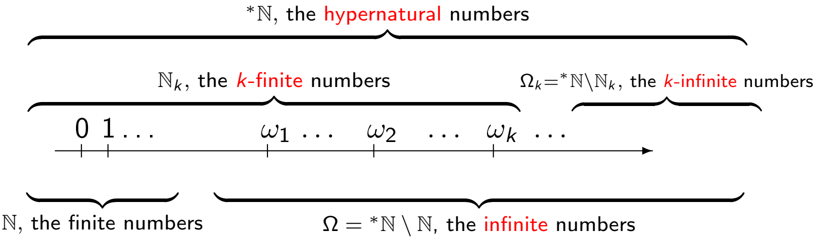

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Thus, to verify whether $k \in A$, we only need to perform finitely many operations (i.e. determine if $\psi(k, M)$).

$\mathbb{NSA}$ has Ω-CA instead of Δ_1-CA.

Principle (Ω-CA)

All Ω-invariant sets exist.

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WHY is this a good/faithful/reasonable/... translation?

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Central: algorithm and proof	Central: Ω-invariance and transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \,\mathbb{V}\, B$: $[A \vee B] \wedge [A \to A \in \mathbb{T}] \wedge [B \to B \in \mathbb{T}]$
$\approx$ "an algo decides if A or if B"	$\approx$ "an Ω-inv.proc. decides if A or if B"
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rrightarrow B$: $A \wedge [A \in \mathbb{T}] \to B \wedge [B \in \mathbb{T}]$
$\neg A$: $A \to (0 = 1)$	$\sim A$: $A \Rrightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: an Ω-inv. proc. computes x_0 such that $A(x_0)$

WHY is this a good/faithful/reasonable/... translation?

BECAUSE the non-algorithmic/non-constructive principles behave the same!

Constructive Reverse Mathematics

Constructive Reverse Mathematics

BISH (based on IL)

non-constructive/non-algorithmic

NSA (based on CL)

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	
$\updownarrow$	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	
$\updownarrow$	
MCT: monotone convergence thm	
$\updownarrow$	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	
$\updownarrow$	
MCT: monotone convergence thm	
$\updownarrow$	
CIT: Cantor intersection thm	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	
$\updownarrow$	
MCT: monotone convergence thm	
$\updownarrow$	
CIT: Cantor intersection thm	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	
$\updownarrow$	
CIT: Cantor intersection thm	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
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LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$	$\updownarrow$
CIT: Cantor intersection thm	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} {\sim}(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$	$\updownarrow$
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\updownarrow$
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim (x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\updownarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
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LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} {\sim}(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\updownarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
(point in intersection computed by algo)	

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbin{\mathbb{V}} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbin{\mathbb{V}} \sim(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\updownarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
(point in intersection computed by algo)	(point in intersection computed by Ω-inv. proc.)

Constructive Reverse Mathematics

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbin{\mathbb{V}} \sim P$
$\updownarrow$	$\updownarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbin{\mathbb{V}} \sim(x < 0))$
$\updownarrow$	$\updownarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\updownarrow$ (limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
	$\updownarrow$
	Universal Transfer
	$(\forall n \in \mathbb{N})\varphi(n) \to (\forall n \in {}^*\mathbb{N})\varphi(n)$

Constructive Reverse Mathematics II

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
LLPO	
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	
$\updownarrow$	

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NSA (based on CL)

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

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NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

NSA (based on CL)

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

NSA (based on CL)

non-Ω-invariant

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

NSA (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$

$\updownarrow$

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

NSA (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \,\mathbb{V} \sim Q$

$\updownarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\updownarrow$

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\updownarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\updownarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\updownarrow$
IVT: Intermediate value theorem	

Constructive Reverse Mathematics II

BISH (based on IL)
non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

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LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

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NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

NSA (based on CL)
non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \,\mathbb{V}\, \sim Q$

$\updownarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\updownarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \,\mathbb{V}\, y = 0)$

$\updownarrow$

$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

(int. value computed by algo)

NSA (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$

$\updownarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$

$\updownarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$

$\updownarrow$

$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics II

BISH (based on IL)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

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LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

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NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

(int. value computed by algo)

NSA (based on CL)

non-Ω-invariant

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \,\mathbb{V} \sim Q$

$\updownarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\updownarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \,\mathbb{V}\, y = 0)$

$\updownarrow$

$\mathbb{IVT}$: Intermediate value theorem

(int. value computed by Ω-inv. proc.)

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\updownarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\updownarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\updownarrow$
IVT: Intermediate value theorem	$\mathbb{IVT}$: Intermediate value theorem
(int. value computed by algo)	(int. value computed by Ω-inv. proc.)
Axioms of $\mathbb{R}$: $\neg(x > 0 \wedge x < 0)$	

Constructive Reverse Mathematics II

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\updownarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\updownarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\updownarrow$
IVT: Intermediate value theorem	$\mathbb{IVT}$: Intermediate value theorem
(int. value computed by algo)	(int. value computed by Ω-inv. proc.)
Axioms of $\mathbb{R}$: $\neg(x > 0 \wedge x < 0)$	Axioms of $\mathbb{R}$: $\sim(x > 0 \wedge x < 0)$

Constructive Reverse Mathematics III

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	
$\updownarrow$	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$

$\updownarrow$

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic MP: For $P \in \Sigma_1$, $\neg\neg P \to P$ $\updownarrow$ MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$ $\updownarrow$ EXT: the extensionality theorem	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \rightarrow P$	
$\updownarrow$	
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \rightarrow x > 0)$	
$\updownarrow$	
EXT: the extensionality theorem	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	
$\updownarrow$	
EXT: the extensionality theorem	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
$\updownarrow$	$\updownarrow$
EXT: the extensionality theorem	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
$\updownarrow$	$\updownarrow$
EXT: the extensionality theorem	$\mathbb{EXT}$: the extensionality theorem

Constructive Reverse Mathematics III

BISH (based on IL) non-constructive/non-algorithmic	NSA (based on CL) non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \rightarrow P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \rightarrow x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
$\updownarrow$	$\updownarrow$
EXT: the extensionality theorem	$\mathbb{EXT}$: the extensionality theorem
WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$	
$\updownarrow$	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \rightarrow P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \rightarrow x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
$\updownarrow$	$\updownarrow$
EXT: the extensionality theorem	$\mathbb{EXT}$: the extensionality theorem
WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$	
$\updownarrow$	
WLPR: $(\forall x \in \mathbb{R})\big[\neg\neg(x > 0) \vee \neg(x > 0)\big]$	
$\updownarrow$	

Constructive Reverse Mathematics III

BISH (based on IL)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \rightarrow P$

$\updownarrow$

MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \rightarrow x > 0)$

$\updownarrow$

EXT: the extensionality theorem

WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$

$\updownarrow$

WLPR: $(\forall x \in \mathbb{R})\big[\neg\neg(x > 0) \vee \neg(x > 0)\big]$

$\updownarrow$

DISC:

A discontinuous $2^{\mathbb{N}} \rightarrow \mathbb{N}$-function exists.

NSA (based on CL)

non-Ω-invariant

$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$

$\updownarrow$

$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$

$\updownarrow$

$\mathbb{EXT}$: the extensionality theorem

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \to P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
$\updownarrow$	$\updownarrow$
EXT: the extensionality theorem	$\mathbb{EXT}$: the extensionality theorem
WLPO: For $P \in \Sigma_1$, $\neg\neg P \vee \neg P$	$\mathbb{WLPO}$: For $P \in \Sigma_1$, $\sim\sim P \,\mathbb{V} \sim P$
$\updownarrow$	$\updownarrow$
WLPR: $(\forall x \in \mathbb{R})\big[\neg\neg(x > 0) \vee \neg(x > 0)\big]$	
$\updownarrow$	
DISC: A discontinuous $2^{\mathbb{N}} \to \mathbb{N}$-function exists.	

Constructive Reverse Mathematics III

BISH (based on IL)	NSA (based on CL)
non-constructive/non-algorithmic	non-Ω-invariant
MP: For $P \in \Sigma_1$, $\neg\neg P \rightarrow P$	$\mathbb{MP}$: For $P \in \Sigma_1$, $\sim\sim P \Rrightarrow P$
$\updownarrow$	$\updownarrow$
MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \rightarrow x > 0)$	$\mathbb{MPR}$: $(\forall x \in \mathbb{R})(\sim\sim(x > 0) \Rrightarrow x > 0)$
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DISC: A discontinuous $2^{\mathbb{N}} \to \mathbb{N}$-function exists.	$\mathbb{DISC}$: A discontinuous $2^{^*\mathbb{N}} \to {}^*\mathbb{N}$-function exists.

Constructive Reverse Mathematics

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Same for WMP, FAN_Δ, BD-N, and $\mathsf{MP}^\vee$.

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WLPO $\rightarrow$ LLPO	$\mathbb{WLPO} \rightarrow \mathbb{LLPO}$
LLPO $\rightarrow$ $\text{MP}^\vee$	$\mathbb{LLPO} \rightarrow \mathbb{MP}^\vee$
LPO $\rightarrow$ BD-N	$\mathbb{LPO} \rightarrow \mathbb{BD}\text{-}\mathbb{N}$
LPO $\rightarrow$ FAN_Δ	$\mathbb{LPO} \rightarrow \mathbb{FAN}_\Delta$

Conclusion

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Reverse-engineering Reverse Mathematics (Fuchino-sensei)

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For Bishop's notion of algorithm, we conclude:

Algorithmic $\approx$ Ω-invariant
because
Non-algorithmic $\approx$ Non-Ω-invariant.

Why is there a connection?

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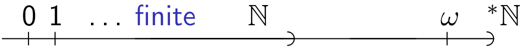

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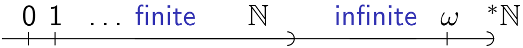

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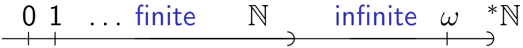

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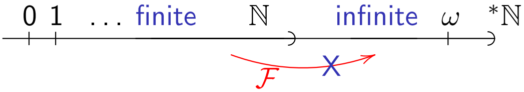

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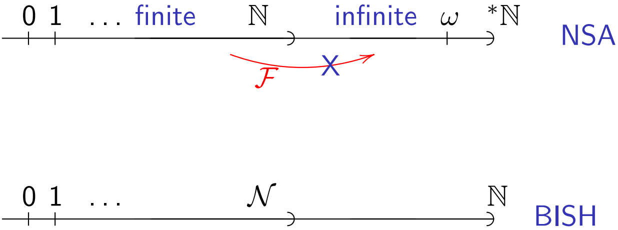

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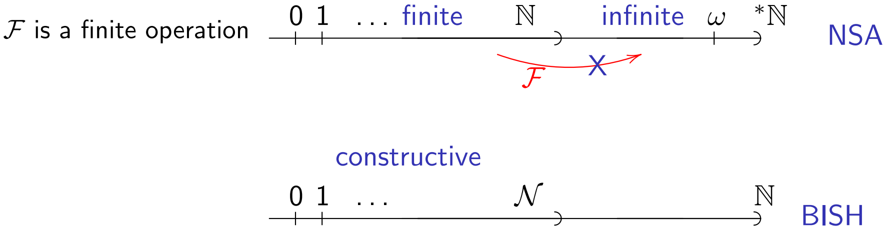

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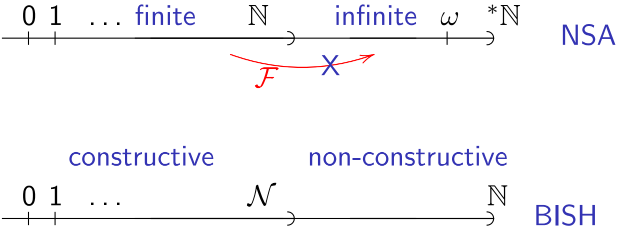

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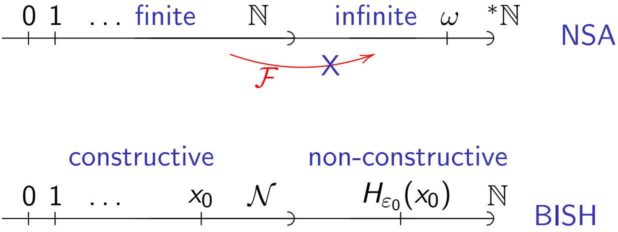

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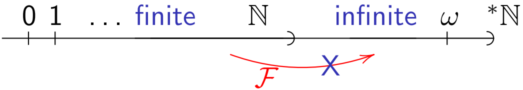

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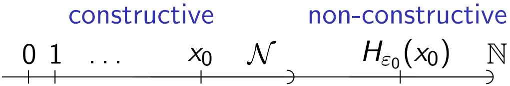

BISH

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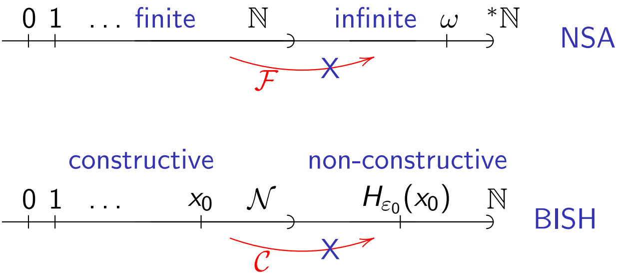

Future research: Bounded Arithmetic

Is Ω-invariance useful for Bounded Arithmetic?

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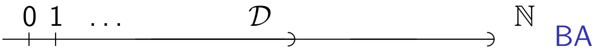

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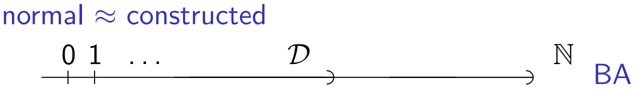

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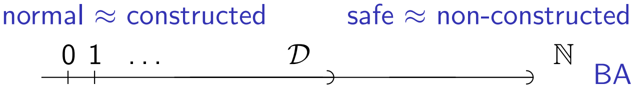

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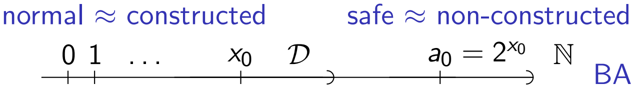

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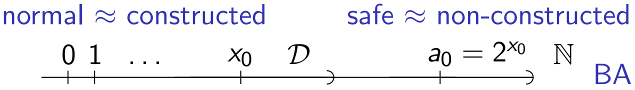

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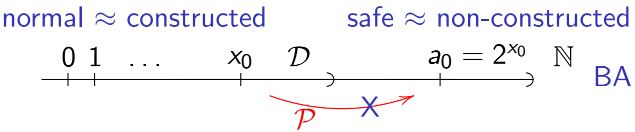

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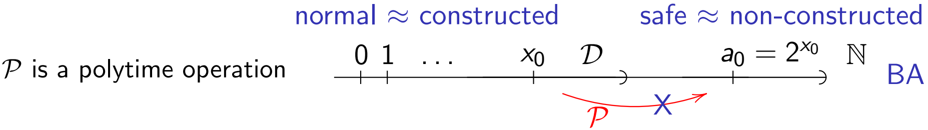

P $\overset{?}{\approx}$ Safe-invariance: $(\forall \vec{x} \in \mathcal{D})(\forall \vec{a}, \vec{b} \notin \mathcal{D})\big[f(\vec{x};\vec{a}) = f(\vec{x};\vec{b})\big]$.

Future work: Type Theory

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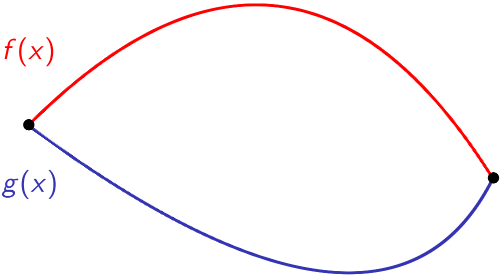

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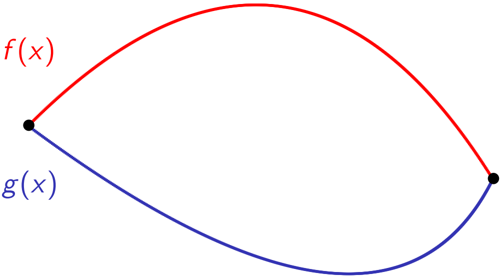

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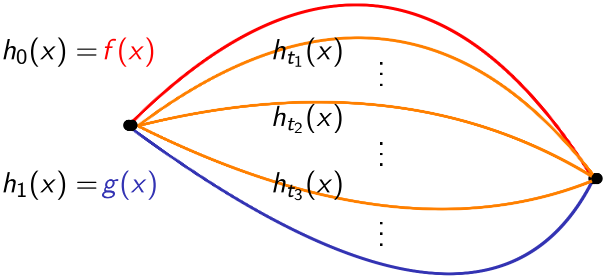

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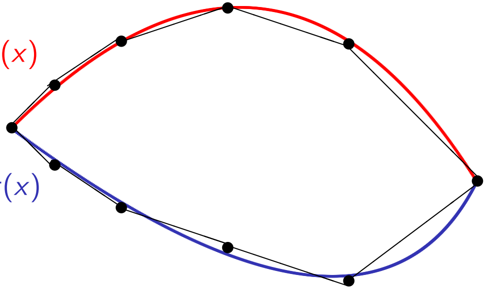

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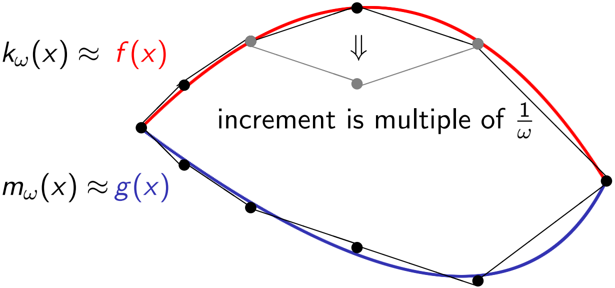

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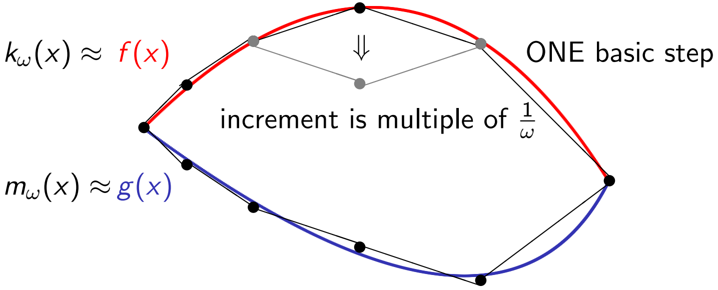

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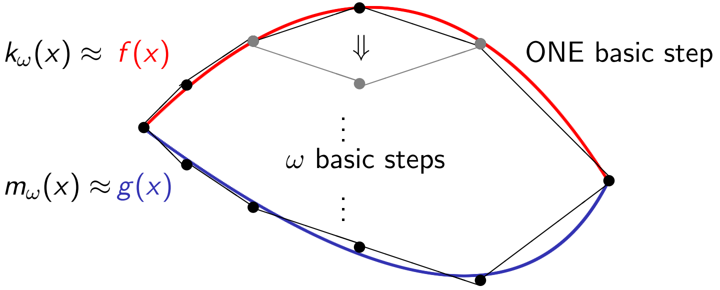

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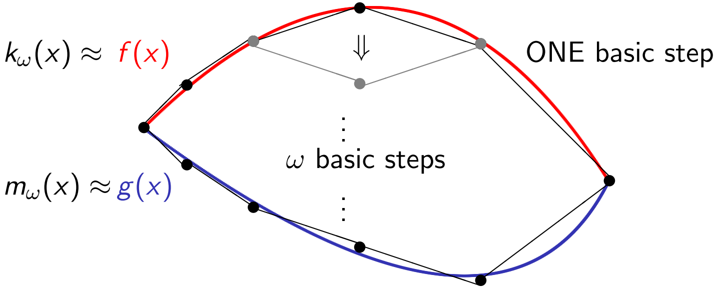

Independent of the choice of ω

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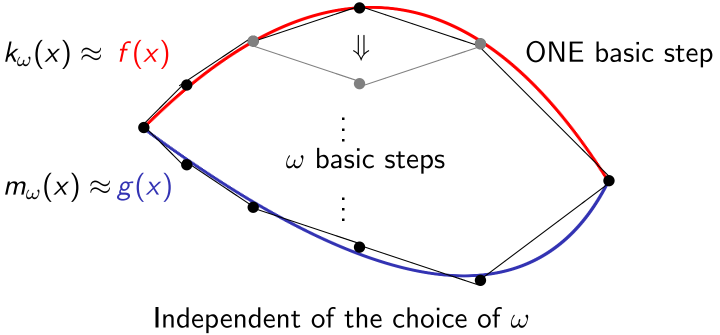
Homotopy: $\approx$ Ω-invariant broken-line transformation $h_{\omega,t}$ of f to g.



On (anti)realism

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VERSUS

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How does X know he is not actually sitting in Y?

Philosophy of Physics

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An end result with physical meaning will not depend on the infinite number/infinitesimal used, i.e. it is Ω-invariant. (Connes)

On Robustness

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Any questions?