The Solovay Hierarchy

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The large cardinal phenomenon

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- Forcing axioms such as *PFA*: used to solve problems in analysis, operator algebras, combinatorics and etc.
- Determinacy axioms such as *PD* or *AD*: used to solve problems in analysis.
- Generic embeddings: used to solve many combinatorial problems.

Reversing the large cardinal phenomenon

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Remark

- *Without such reversals the large cardinal phenomenon has interesting but ultimately not an important content.*
- *However, such reversals have been established for a very small initial segment of the large cardinal hierarchy.*

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Remark

There is no known systematic way of getting reversals much beyond the large cardinal axiom of the theorem.

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Theorem (Kunen)

There is no $j : V \to V$ such that $j \neq id$.

Examples

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- κ is a measurable cardinal if there is $j : V \to M$ such that $crit(j) = \kappa$ and M is closed under κ-sequence, i.e. for every $f : \kappa \to M$, $f \in M$.
- κ is a supercompact cardinal if for every λ there is $j : V \to M$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Examples

1. measurable cardinals,
2. strong cardinals,
3. Woodin cardinals,
4. Shelah cardinals,
5. superstrong cardinals,
6. subcompcact cardinals,
7. supercompact cardinals,
8. huge cardinals,
9. etc (look at Kanamori's book).

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8. huge cardinals,
9. etc (look at Kanamori's book).

Remark

Woodin cardinals are tiny when compared to superstrong cardinals which are tiny when compared to supercompact cardinals.

Conjecture (The PFA Conjecture)

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Remark

- *Part 3 is the most important one as it will give an equiconsistency.*
- *While the conjecture has been open for a long time, it is only a test question.*

The classical approach: the inner model problem

Problem (The Inner Model Problem)

Construct canonical inner models with large cardinals.

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- *The canonical inner models are models that resemble L, such models are called mice.*
- *While the problem is open for almost all large cardinals that are significantly bigger than Woodin cardinals, the desired cardinal is the supercompact cardinal.*
- *The goal is to develop tools for systematically constructing such canonical models with large cardinals while working under various theories extending ZFC.*

The origin of the problem

Definition (Gödel)

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \{A \subseteq L_\alpha : A$ is definable over $(L_\alpha, \in)$ with parameters$\}$.
- for limit λ, $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.
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Theorem (Scott, 1961)

Suppose there is a measurable cardinal. Then $V \neq L$.

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- Because of this, it is impossible not to ask whether large cardinals can coexist with such a canonical structure.
- This is exactly the content of the inner model problem.
- But what are these canonical models?

The idea.

Remark

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- *All large cardinals can be defined in terms of the existence of ultrafilters or extenders.*
- *An extender E is a coherent family of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings than usual ultrafilters.*
- *Since all large cardinals can be defined via extenders, it is natural to look for canonical models with large cardinals among the models of the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.*

The model $L[A]$

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Remark

- *$L[A]$ may not be canonical, it depends on A.*
- *The idea is to consider $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders and show that it is "canonical" and has large cardinals.*

Premice and mice

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- A mouse is an iterable premouse.
- Iterability is a fancy way of saying that all the ways of taking ultrapowers and direct limits produce well-founded models. More precisely, look at the picture.

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3. In general, to have a good theory of mice, $\omega_1 + 1$-iterability is all that is needed.
4. Notice that it must be hard to construct such strategies as there are trees of height ω_1 with no branch.

The inner model problem revisited.

Problem (The inner model problem)

Construct mice with large cardinals.

Mice are canonical: comparison

Definition

- Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \trianglelefteq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \restriction \alpha$.

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- Comparison is the statement: Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies Σ and Λ, there are a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$ and a Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$.

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Theorem (Mitchell-Steel, early 90s)

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Corollary

If $\mathcal{M}$ and $\mathcal{N}$ are two mice then $\mathbb{R}^2 \cap \leq^{\mathcal{M}}$ is compatible with $\mathbb{R}^2 \cap \leq^{\mathcal{N}}$.

Remark

Hence, mice can only have canonical reals.

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- *There is a recent approach that goes through descriptive set theory.*
- *The classical approach, via K^c-constructions, reduces to constructing canonical iteration strategies, or $\omega_1 + 1$-iteration strategies whose ω_1 part is universally Baire. This approach, too, seems to lead to descriptive set theory.*

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- Key Point: For this to be successful, it is necessary to show that the Solovay hierarchy, just like the large cardinal hierarchy, is a consistency strength hierarchy that covers all the levels of the large cardinal hierarchy.

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- Key Point: For this to be successful, it is necessary to show that the Solovay hierarchy, just like the large cardinal hierarchy, is a consistency strength hierarchy that covers all the levels of the large cardinal hierarchy. *This has not yet been established.*

The main problem of descriptive inner model theory

Problem (The main problem)

Find determinacy theories that catch up with the large cardinal hierarchy.

Remark

The Solovay hierarchy is one candidate.

The Solovay sequence

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4. $\Theta = \theta_\Omega$.

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Theorem (Martin,Woodin, 80s)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $AD_{\mathbb{R}}$ implies that $\Theta = \theta_\Omega$ for some limit ordinal Ω.

Some important axioms from the hierarchy

HOD is the class of hereditarily ordinal definable sets. It satisfies *ZFC*.

Examples

- $AD_{\mathbb{R}}$ + "Θ is regular".
- $AD_{\mathbb{R}}$ + "Θ is Mahlo in HOD".
- $AD_{\mathbb{R}}$ + "Θ is weakly compact in HOD".
- $AD_{\mathbb{R}}$ + "Θ is measurable".
- $AD_{\mathbb{R}}$ + "Θ is Mahlo".

More important axioms

- A set of reals is called κ-Suslin if there is a tree $T \subseteq \cup_{n<\omega}\omega^n \times \kappa^n$ such that

$$A = \{x \in \omega^\omega : \exists f \in \kappa^\omega((x, f) \text{ is a branch of } T)\}.$$

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- (LST) $AD^+ + \Theta = \theta_{\alpha+1} +$ "θ_α is the largest Suslin cardinal".
- Let ϕ be a large cardinal axiom. Then let
$$S_\phi =_{def} LST + V_\Theta^{\mathrm{HOD}} \vDash \exists\kappa\phi[\kappa].$$

The main conjecture

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For each ϕ, S_ϕ is consistent relative to some large cardinal.

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- *(Woodin) Under AD, if $\theta_{\alpha+1}$ exists then it is Woodin in HOD.*
- *One arrives at these axioms by analyzing HOD: Under AD^+, HOD is a some kind of mouse, a hod mouse, a structure constructed from a sequence of extenders and strategies. The analysis implies that we ought to consider such axioms.*

The consistency of the axioms.

Theorem (2008)

Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model M such that $\mathbb{R} \subseteq M$ *and* $M \models AD_{\mathbb{R}} +$ *"*Θ *is measurable".*

The consistency of the axioms.

Theorem (2008)

Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model M such that $\mathbb{R} \subseteq M$ *and* $M \models AD_{\mathbb{R}} +$ *"*Θ *is measurable".*

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- *Many similar axioms from the Solovay hierarchy can be shown to be consistent relative to some large cardinal axiom. In particular, many approximations of LST have been shown to be consistent relative to large cardinals.*

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- *However, LST itself is somewhat mysterious, perhaps for a good reason.*

Examples of reversals using the Solovay hierarchy

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Assume PFA. Then there is an inner model M such that $\mathbb{R} \subseteq M$ *and* $M \vDash AD_{\mathbb{R}} +$ *"*Θ *is regular".*

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Corollary

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Remark

The usual forcing methods require at least a supercompact cardinal to force either of the conclusions and both of these conclusions have a significant large cardinal strength and are probably equiconsistent with $AD_{\mathbb{R}}$ + *"*Θ *is regular".*

Forcing failure of square

Theorem (Caicedo, Larson, S., Schindler, Steel, Zeman, 2011)

Assume $AD_{\mathbb{R}}$. Suppose the set

$$\{\kappa < \Theta : \kappa \textit{ is regular in } \mathrm{HOD} \textit{ and } \mathrm{cf}(\kappa) = \omega_1\}$$

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Remark

To force just $\neg\Box(\omega_2) + \neg\Box_{\omega_2}$ via conventional techniques one needs at least a subcompact cardinal which is much stronger than superstrong cardinals.

An alternative way of solving the main open problem is the following.

Problem

Find a determinacy theory T such that the following hold.

1. *T implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces $ZFC + \neg\Box(\omega_3) + \neg\Box_{\omega_3}$.*

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Remark

Letting T be as above, there is a good evidence that it will have to be stronger than a superstrong cardinal.

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- The analysis of HOD might just as well lead to, via Woodin's axiom, the theory of ultimate L or rather, the ultimate foundation appropriate for studying all of mathematics without any bias towards a particular theory.
- For now, however, we can only say: to be continued.