Ramsey classes of finite trees and SOP_2

Lynn Scow

University of Illinois at Chicago

ASL 2012 North American Annual Meeting

Outline

1. basic notions
2. indiscernibility as a tool
3. application to trees

dividing lines

- Classification theory seeks to isolate properties that act as good dividing lines between more-complicated and less-complicated theories.
- Often such a property is described by the presence of a formula encoding certain information.
- In our discussion of trees T, nodes $\eta, \nu \in T$ will be written $\eta \perp \nu$ to signify that they are incomparable with respect to the partial tree order.
- Typically T will be $\omega^{<\omega}, 2^{<\omega}$.
- In general a, x stand for finite tuples $\overline{a}, \overline{x}$ of parameters/variables.

TP_1

- Here is such a dividing-line property.

Definition

A theory T has *tree property-1 (TP_1)* if there is a model $M \vDash T$, a formula $\varphi(x; y)$ and parameters $a_\eta \in |M|$ such that:

1. $\{\varphi(x; a_{\sigma\upharpoonright n}) : \sigma \in \omega^\omega\}$ is consistent ("branches are consistent"), and
2. $\{\varphi(x; a_\eta) \wedge \varphi(x; a_\nu)\}$ is inconsistent, for $\eta, \nu \in \omega^{<\omega}, \eta \perp \nu$ ("incomparable nodes are inconsistent")

- A theory with TP_1 is on the more-complicated side of the dividing line provided by the property, TP_1.
- Naming a set $\omega^{<\omega}$ implies facts about this set that can be expressed in a first-order way. Can we isolate the relevant parts of the "theory" of this set?

TP_1 and SOP_2

- Here we name a second property, SOP_2 which is equivalent to TP_1 for theories:

Definition

A theory T has *strong order property-2 (SOP_2)* if there is a model $M \vDash T$, a formula $\varphi(x; y)$ and parameters $a_\eta \in |M|$ such that:

1. $\{\varphi(x; a_{\sigma \restriction n}) : \sigma \in 2^\omega\}$ is consistent ("branches are consistent"), and
2. $\{\varphi(x; a_\eta) \wedge \varphi(x; a_\nu)\}$ is inconsistent, for $\eta, \nu \in 2^{<\omega}, \eta \perp \nu$ ("incomparable nodes are inconsistent")

- There are many relations we could suggest to be basic relations on our tree: $\trianglelefteq$ (partial order), $\wedge$ (meet function), $<_{\mathrm{lex}}$ (linear order extending $\trianglelefteq$).
- We need only look at $\trianglelefteq$-embeddings to transfer SOP_2 to TP_1; to obtain *trees* with the right partition properties, we may be required to take on more of the language.

what structure on $2^{<\omega}$ is relevant to SOP_2?

- We might feel we had isolated the relevant part of the "theory" of $2^{<\omega}$ if somehow $M = (2^{<\omega}, \trianglelefteq)$ and $\varphi(x; y) = (x \trianglelefteq y)$ gave the most canonical example of SOP_2. (This is not so.)
- The *strict order property (sOP)* is another dividing-line property that is known to be strictly stronger than SOP_2.
- A theory T has the strict order property if there is a formula $\varphi(x; y)$ and parameters in some $M \vDash T$, $(a_i : i < \omega)$ such that the following implication holds strictly:

$$\varphi(x, a_i) \Rightarrow \varphi(x; a_{i+1})$$

- $x \trianglelefteq y$ witnesses the sOP in $2^{<\omega}$, so this can't be our best example of SOP_2

indiscernibles

- An early effort to better understand the witnesses $(a_\eta : \eta \in 2^{<\omega})$ to SOP_2 in a theory was to find an assumption of *indiscernibility* we could make, without loss of generality.
- This approach was first pursued in [DS04] for SOP_2; the following notion of I-indexed indiscernible is from [She90]:

Definition

Fix structures I, M. An I-indexed indiscernible is a set of parameters from M, $(b_i : i \in I)$ such that for all $n < \omega$ and $i_1, \ldots, i_n; j_1, \ldots, j_n$ from I:

$$\mathrm{qftp}(i_1, \ldots, i_n; I) = \mathrm{qftp}(j_1, \ldots, j_n; I) \Rightarrow \mathrm{tp}(b_{i_1}, \ldots, b_{i_n}) = \mathrm{tp}(b_{j_1}, \ldots, b_{j_n})$$

- We say "quantifier-free type" in order to get a stronger notion of homogeneity.

the uniform data in a set of parameters

- We would like to assume parameters "in a certain configuration" are indiscernible, without loss of generality.

Definition

Fix a structure I and parameters $\mathbf{I} := (a_i : i \in I)$ from some structure M. Define the *EM-type of* $\mathbf{I}$ to be:

$$\mathrm{EMtp}(\mathbf{I})(\{x_i : i \in I\}) := \{\psi(x_{i_1}, \ldots, x_{i_n}) : n < \omega, \psi(x_1, \ldots, x_n) \in \mathcal{L}(M),$$

$$\text{for all } j_1, \ldots, j_n \text{ from } I \text{ such that } \mathrm{qftp}(\bar{j}) = \mathrm{qftp}(\bar{i}),$$

$$\vDash \psi(a_{j_1}, \ldots, a_{j_n})\}$$

- The $I = (\omega, <)$ case of the above is referred to as EM($\mathbf{I}$) in [TZ11]. We are careful not to confuse our terminology with EM(I,Φ) ([Bal09, She90]), which is a term that denotes a certain type of model. Note that EMtp($\mathbf{I}$) derives a kind of profile/pattern/template from an I-indexed set of parameters, whether or not this set is indiscernible.

age and Ramsey class

- We want some terminology for the next development. Fix a structure I (with some intended language.)
- The *age*, $\text{age}(I)$, of a structure I is the class of all finitely-generated substructures of I, closed under isomorphism.
- Let $\binom{C}{A}$ be the substructures of C isomorphic to A (the "A-substructures of C.")
- Say that a class $\mathcal{K}$ of finite structures is a *Ramsey class* if for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ such that given any 2-coloring $c : \binom{C}{A} \to \{0, 1\}$ there is a $B' \subseteq C$, $B' \cong B$, such that $c : \binom{B'}{A} \to \{i_0\}$, for some choice of $i_0 \in \{0, 1\}$.
- It is equivalent to state the property for k-colorings, where $k < \omega$ is arbitrary ≥ 2.

modeling the uniform data

- Consider the property: for any I-indexed parameters $\mathbf{I} = (a_i : i \in I)$ from sufficiently-saturated M we may find I-indexed indiscernible $\mathbf{J} = (b_s : s \in I) \models \mathrm{EMtp}(\mathbf{I})$.
- We may call the latter the *modeling property* for I-indexed indiscernibles.

Theorem ([Sco12])

For I a structure in a finite relational language, where one basic relation $<$ linearly orders I, I-indexed indiscernibles have the modeling property just in case age(I) is a Ramsey class.

functions in the index structure

- The following generalization helps us deal with the case of $I = (2^{<\omega}, \trianglelefteq, \wedge, <_{\mathrm{lex}})$.

Theorem

For uniformly locally finite I in a finite language, where one basic relation $<$ linearly orders I, I-indexed indiscernibles have the modeling property just in case age(I) is a Ramsey class.

- A similar argument to one in [Sco12] shows that the modeling property implies the Ramsey class property for age(I).
- This argument requires that we isolate the quantifier-free types by way of formulas, and we can still do this.

RC $\Rightarrow$ modeling property

- This direction is a little harder because there isn't as obvious a correspondence between realizations of a quantifier-free type and substructures of I.
- For $\bar{i} \models \eta(v_1, \ldots, v_n)$, a complete quantifier-free type (consistent with $v_1 < \ldots < v_n$), and $A = \langle \bar{i} \rangle$ the substructure generated by $\bar{i}$, let $\mathrm{cl}(\bar{i})(x_1, \ldots, x_N)$ be the isomorphism-type of A in $<$-increasing enumeration.
- Let $x_{i_1}, \ldots, x_{i_n}$ be the indices at which $\bar{i}$ occurs in the increasing enumeration of A. Every copy of A determines a unique copy of $\bar{i}$, and every copy of $\bar{i}$ in a structure B occurs within a copy of A in B.
- Homogeneity for copies of A implies homogeneity for $\bar{j} \models \eta$, as we shall see from the nature of a *type-coloring*:

type-colorings

- For a finite structure B of size m, let $p_B(x_1, \ldots, x_m)$ be the complete quantifier-free type of B listed in $<$-increasing order.

Definition

Let I be any structure. By a *type-coloring of tuples from I* we mean a χ-coloring (χ a cardinal)

$$c : I^{<\omega} \to \chi$$

with the property that for length-m $\bar{b}, \bar{b}' \in I$ such that $c(\bar{b}) = c(\bar{b}')$, for any $n \leq m$

$$c(\langle b_{i_1}, \ldots, b_{i_n} \rangle) = c(\langle b'_{i_1}, \ldots, b'_{i_n} \rangle)$$

- If we let $\Delta(x_1, \ldots, x_n)$ be a finite set of formulas from M, then an I-indexed set in M, $(a_i : i \in I)$ comes equipped with a (finite) type-coloring by way of $c(\langle i_1, \ldots, i_n \rangle) = \mathrm{tp}_\Delta(a_{i_1}, \ldots, a_{i_n}; M)$.

in sum

- Given an I-indexed set of parameters $\mathbf{I} = (a_i : i \in I)$, we have a type-coloring of tuples from I.
- Here is the "type of our indiscernible":
  $\Gamma(x_i : i \in I) = \{\psi(x_{i_1}, \ldots, x_{i_m}) \to \psi(x_{j_1}, \ldots, x_{j_m}) :$
  $$\psi(x_1, \ldots, x_m) \in \mathcal{L}(M);\ \text{qftp}(\bar{i}) = \text{qftp}(\bar{j});\ \bar{i}, \bar{j} \text{ from } I\}$$
- To find our I-indexed indiscernible $\vDash \text{EMtp}(\mathbf{I})$, it suffices to satisfy a finite portion of the "type of our indiscernible" in $(a_i : i \in I)$, a portion indexed by a finite set $I_0 \subseteq I$ and mentioning a finite set of $\mathcal{L}(M)$-formulas Δ.
- This amounts to, for given structures $A, B = \langle I_0 \rangle$, finding a homogeneous $B' \cong B$ in I for the type-coloring above, as it applies to A-substructures of I.
- In general we must perform an induction on the $A_1, \ldots, A_n$ that are generated by tuples from I_0.

restrictions

- It would be good to develop a technology for countable languages.
- The non-locally finite case does not seem practicable, because partition properties often fail when we are searching for an infinite substructure B.
- For example, $\mathbb{Q} \nrightarrow (\mathbb{Q})_2^{\{a_1 < a_2\}}$.
- Similarly for the random graph $\mathcal{R}$: $\mathcal{R} \nrightarrow (\mathcal{R})_2^{\{a_1 R a_2\}}$.

Thanks

Thanks for your attention!

John T. Baldwin.
Categoricity, volume 50 of *University Lecture Series*.
American Mathematical Society, Providence, RI, 2009.

M. Džamonja and S. Shelah.
On $\triangleleft^*$-maximality.
Annals of Pure and Applied Logic, 125(1-3):119–158, 2004.

L. Scow.
Characterization of NIP theories by ordered graph-indiscernibles, 2012.

10.1016/j.apal.2011.12.013.

S. Shelah.
Classification Theory and the number of non-isomorphic models (revised edition).
North-Holland, Amsterdam-New York, 1990.

K. Tent and M. Ziegler.
A Course in Model Theory, 2011.
to appear in Cambridge University Press, ASL Lecture Notes in Logic.