Presenting the effectively closed Medvedev degrees requires $\mathbf{0}'''$

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ASL 2012 North American Annual Meeting
Madison, WI
March 31, 2012

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The mass problem $\mathcal{A}$ is effectively closed if $\mathcal{A} = [T]$ for some computable tree $T \subseteq 2^{<\omega}$. Equivalently, $\mathcal{A}$ is effectively closed if it is a Π^0_1 class.

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$\mathbf{1} = \deg_s$(complete consistent extensions of *PA*).

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$\mathcal{D}_\mathrm{s}$ is a Brouwer algebra (for every a and b there is a least c with $a + c \geq b$).

Neither $\mathcal{D}_\mathrm{s,cl}$ (Lewis, Shore, Sorbi) nor $\mathcal{E}_\mathrm{s}$ (Higuchi) is a Brouwer algebra.

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Theorem (S)

- $\mathrm{Th}(\mathcal{D}_{\mathrm{s}}) \equiv_1 \mathrm{Th}_3(\mathcal{N})$ *(independently by Lewis, Nies, & Sorbi).*
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Today's theorem:

Theorem (S)

The degree of $\mathcal{E}_{\mathsf{s}}$ is $\mathbf{0}'''$. That is, $\mathbf{0}'''$ computes a presentation of $\mathcal{E}_{\mathsf{s}}$, and every presentation of $\mathcal{E}_{\mathsf{s}}$ computes $\mathbf{0}'''$.

Presentations of $\mathcal{E}_\mathsf{s}$

Definition

A presentation of $\mathcal{E}_\mathsf{s}$ is a pair of functions $+, \times \colon \omega \times \omega \to \omega$ such that the structure $(\omega; +, \times)$ is isomorphic to $\mathcal{E}_\mathsf{s}$. The degree of a presentation is $\deg_\mathsf{T}(+ \oplus \times)$.

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That $\mathbf{0}'''$ computes a presentation follows from the fact that the relation $[T_i] \leq_\mathrm{s} [T_j]$ (where T_i and T_j are primitive recursive subtrees of $2^{<\omega}$ with indices i and j) is a Σ^0_3 property of $\langle i, j\rangle$:

$$[T_i] \leq_\mathrm{s} [T_j] \Leftrightarrow \exists e \forall n \exists s (\forall \sigma \in 2^s)(\sigma \in T_j \to \Phi_e(\sigma) \restriction n \in T_i)$$

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So we need to prove that every presentation of $\mathcal{E}_s$ computes $\mathbf{0}'''$.

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An r.e. separating class is a mass problem of the form

$$\mathcal{S}(A, B) = \{C \in 2^\omega \mid A \subseteq C \subseteq B^c\}$$

for disjoint r.e. sets A and B. An r.e. separating degree is the Medvedev degree of an r.e. separating class.

Spines

Definition

Let $\mathcal{Q}$ be a Π^0_1 class with no recursive member. Let $\{\sigma_n\}_{n\in\omega}$ be a recursive sequence of pairwise incomparable strings such that $\bigcup_{n\in\omega} I(\sigma_n) = 2^\omega \setminus \mathcal{Q}$. Let $\{\mathcal{S}_n\}_{n\in\omega}$ be a recursive sequence of Π^0_1 classes. Then define

$$\mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in\omega}) = \mathcal{Q} \cup \bigcup_{n\in\omega} \sigma_n{}^\frown \mathcal{S}_n.$$

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Lemma

Let $\{\mathcal{Q}\} \cup \{\mathcal{S}_n\}_{n\in\omega}$ be a recursive sequence of r.e. separating classes that is an $\leq_s$-antichain, and let

$$\mathbf{w} = \deg_s(\text{spine}(\mathcal{Q}, \{\mathcal{S}_n\}_{n\in\omega})).$$

If $\mathbf{x}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_s \deg_s(\mathcal{S}_n)$ for some n.

Coding parameters

Let $\mathcal{Q}$, $\{\mathcal{S}_{0,i}\}_{i\in\omega}$, and $\{\mathcal{S}_{1,i}\}_{i\in\omega}$ be such that $\mathcal{Q} \cup \{\mathcal{S}_{0,i}\}_{i\in\omega} \cup \{\mathcal{S}_{1,i}\}_{i\in\omega}$ is a strongly independent recursive sequence of r.e. separating classes. Then let

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$$\begin{aligned}
\mathbf{w}_0 &= \deg_s(\mathcal{W}_0) &\text{for}\quad \mathcal{W}_0 &= \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n}\}_{n\in\omega});\\
\mathbf{w}_1 &= \deg_s(\mathcal{W}_1) &\text{for}\quad \mathcal{W}_1 &= \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{1,n}\}_{n\in\omega});\\
\mathbf{m} &= \deg_s(\mathcal{M}) &\text{for}\quad \mathcal{M} &= \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n} + \mathcal{S}_{1,n}\}_{n\in\omega});\\
\mathbf{p} &= \deg_s(\mathcal{P}) &\text{for}\quad \mathcal{P} &= \mathrm{spine}(\mathcal{Q}, \{\mathcal{S}_{0,n} + \mathcal{S}_{1,n+1}\}_{n\in\omega});\\
\mathbf{v} &= \deg_s(\mathcal{V}) &\text{for}\quad \mathcal{V} &= \sum_{n\in\omega} S_{0,n};\\
\mathbf{r} &= \deg_s(\mathcal{R}) &\text{for}\quad \mathcal{R} &= \mathrm{spine}(\mathcal{Q}, \{\mathcal{R}_n\}_{n\in\omega}),\\
&&&\text{where } \mathcal{R}_n = \sum_{m\neq n} \mathcal{S}_{0,m}.
\end{aligned}$$

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$$D = \{e \mid \exists n(n \in C \wedge \mathcal{Z}_e \leq_{\mathrm{s}} \mathcal{S}_{0,n} \wedge \mathcal{V} \leq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_n)\}.$$

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Lemma

Let $\mathcal{Q}$ be an r.e. separating class, and let $\varphi(e, m, k, \ell)$ be a recursive predicate. Then there is a recursive sequence of Π^0_1 classes $\{\mathcal{X}_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ such that for all $\langle e, m\rangle \in \omega$

$$\deg_s(\mathcal{X}_{\langle e,m\rangle}) = \begin{cases} \mathbf{0} & \text{if } \forall k \exists \ell \varphi(e, m, k, \ell) \\ \deg_s(\mathcal{Q}) & \text{if } \exists k \forall \ell \neg\varphi(e, m, k, \ell). \end{cases}$$

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Let $\mathbf{x} = \deg_{\mathrm{s}}(\mathcal{X})$ for $\mathcal{X} = \mathrm{spine}(\mathcal{Q}, \{\mathcal{Z}_e + \mathcal{X}_{\langle e,m\rangle}\}_{\langle e,m\rangle \in \omega})$.

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Key property: If $\mathcal{Z}_e \leq_s \mathcal{S}_{0,n}$ and $\mathcal{V} \leq_s \mathcal{Z}_e + \mathcal{R}_n$, then $\mathcal{Z}_e \geq_s \mathcal{X}$ iff $n \in C$.

Decoding step 1: walk up to $\deg_s(\mathcal{S}_{0,n})$

To determine if $n \in C$, we first identify a degree $\mathbf{z}_{0,n}$ that is "close" to $\deg_s(\mathcal{S}_{0,n})$:

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(i) each $\mathbf{z}_{i,j}$ meets to $\mathbf{w}_i$;

(ii) each $\mathbf{z}_{0,j} + \mathbf{z}_{1,j} \geq_s \mathbf{m}$;

(iii) each $\mathbf{z}_{0,j-1} + \mathbf{z}_{1,j} \geq_s \mathbf{p}$;

(iv) $\mathbf{y}$ meets to $\mathbf{r}$;

(v) $\mathbf{z}_{0,n} + \mathbf{y} \geq_s \mathbf{v}$.

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(iv) $\mathbf{y}$ meets to $\mathbf{r}$;

(v) $\mathbf{z}_{0,n} + \mathbf{y} \geq_s \mathbf{v}$.

The witnesses will eventually be found because $\mathbf{z}_{i,j} = \deg_s(\mathcal{S}_{i,j})$ and $\mathbf{y} = \deg_s(\mathcal{R}_n)$ satisfy (i)-(v).

Decoding step 2: check if $\mathbf{z}_{0,n} \geq_s \mathbf{x}$

Output "yes" if $\mathbf{z}_{0,n} \geq_s \mathbf{x}$, and output "no" otherwise.

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The proof is by induction, using the facts that the $\mathcal{S}_{i,j}$ are strongly independent, that any $\mathbf{z}$ that meets to $\mathbf{w}_i$ is $\leq_s \deg_s(\mathcal{S}_{i,j})$ for some j, and that any $\mathbf{y}$ that meets to $\mathbf{r}$ is $\leq_s \deg_s(\mathcal{R}_m)$ for some m.

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Therefore $\mathbf{z}_{0,n} = \deg_{\mathrm{s}}(\mathcal{Z}_e)$ where $\mathcal{Z}_e \leq_{\mathrm{s}} \mathcal{S}_{0,n}$ and $\mathcal{V} \leq_{\mathrm{s}} \mathcal{Z}_e + \mathcal{R}_n$.

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Now recall the key property of $\mathcal{X}$: If $\mathcal{Z}_e \leq_s \mathcal{S}_{0,n}$ and $\mathcal{V} \leq_s \mathcal{Z}_e + \mathcal{R}_n$, then $\mathcal{Z}_e \geq_s \mathcal{X}$ iff $n \in C$.

Thank you!

Thanks for coming to my talk!
Do you have any questions about it?