Diagonal extender based Prikry forcing

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Cardinal arithmetic and the exponential operation

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- **The Singular Cardinal Hypothesis (SCH):** if κ is strong limit, then $2^\kappa = \kappa^+$.

The Singular Cardinal Problem: Describe a complete set of rules for the behavior of the exponential function $\kappa \mapsto 2^\kappa$ for singular cardinals κ.

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- Gitik and Woodin significantly reduced the large cardinal hypothesis to a measurable cardinal κ of Mitchell order κ^{++}. This hypothesis was shown to be optimal by Gitik and Mitchell using core model theory.
- So, the failure of SCH is equiconsistent with the existence of a measurable κ of Mitchell order κ^{++}.

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- (Shelah) If $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.
- It is open if the bound can be improved.

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- **Classical Prikry:** starts with a normal measure on κ and adds a cofinal ω-sequence in κ, while preserving cardinals.
- **Violating SCH:** Let κ be a Laver indestructible supercompact cardinal. Force to add κ^{++} many subsets of κ. Then force with Prikry forcing to make κ have cofinality ω. In the final model cardinals are preserved, κ remains strong limit, and $2^\kappa > \kappa^+$. I.e. SCH fails at κ.

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The strategy: add subsets to a large cardinal, then singularize it.

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- No need to add subsets of κ in advance, so can keep GCH below κ (as opposed to the above forcings).
- Allows more flexibility when interleaving collapses in order to make κ a small cardinal (e.g. $\aleph_\omega$).

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5. for $a \subset \kappa^{+\omega+2}$, with $|a| < \kappa$, there are unboundedly many $\beta \in \kappa^{+\omega+2}$, such that for all $\alpha \in a$, $\alpha \leq_E \beta$.

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The hybrid Prikry - the basic modules

$\mathbb{Q} = \mathbb{Q}_0 \cup \mathbb{Q}_1$ is defined as follows:

- $\mathbb{Q}_1 = \{f : \kappa^{+\omega+2} \to \kappa \mid |f| < \kappa^{+\omega+1}\}$
- $\mathbb{Q}_0$ has conditions of the form $p = \langle a, A, f \rangle$ such that:
  - $f \in \mathbb{Q}_1$
  - $a \subset \kappa^{+\omega+2}$, $|a| < \kappa$, and $a \cap \mathrm{dom}(f) = \emptyset$
  - a has an $\leq_E -$ maximal element and $A \in E_{\max a}$
  - for all $\alpha \leq \beta \leq_E \gamma$ in a, $\nu \in \pi_{\max a, \gamma}{}'' A$, $\pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$.
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Conditions in $\mathbb{P}$ are of the form

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   - F_n is a function with domain A_n, for $y \in A_n$, $F_n(y) \in \mathbb{Q}_0$.
3. For $x \in A_n$, denote $F_n(x) = \langle a^n_x, A^n_x, f^n_x\rangle$. Then for $l \leq n < m$, $y \in A_n, z \in A_m$ with $y \prec z$, we have $a^n_y \subset a^m_z$.

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- $\mathbb{P}$ blows up the powerset of κ to $(\kappa^{+\omega+2})^V$. And so, in the generic extension SCH fails at κ.

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We can also interleave collapses in the usual way to make $\kappa = \aleph_\omega$

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  - Or where the tree property holds simultaneously at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$?