A dense family of finite 1-generated distributive groupoids[1]

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ASL North American Annual Meeting
3 April 2012

[1] Portions of this project were supported by a summer fellowship at the Fields Institute, Toronto, ON

Distributive Groupoids

A groupoid $\boldsymbol{G} = \langle G; *\rangle$ is (left)-distributive if

$$\boldsymbol{G} \models \forall xyz\ x * (y * z) \approx (x * y) * (x * z)$$

The class of distributive groupoids will be denoted $\mathcal{LD}$.

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Example

Material implication $\Rightarrow$

$$A \Rightarrow (B \Rightarrow C) \equiv (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$$

is a distributive operation on $\{T, F\}$.

HSP and Free groupoids

$\mathcal{LD}$ is a variety (an equational class), so is closed under taking substructures, direct products, and (surjective) homomorphic images; perhaps most importantly, we are guaranteed free algebras $\boldsymbol{F}_{\mathcal{LD}}(\kappa)$ for all cardinals κ.

The current investigation focusews on 1-generated LD-groupoids (MLDs), i.e. quotients of $\boldsymbol{F}_{\mathcal{LD}}(1)$.

Some Set Theory

Consider the following very strong large cardinal axiom:

There exists $\alpha \in ORD$ and a nontrivial elementary embedding

$$V_\alpha \xrightarrow{j} V_\alpha \tag{R2R}$$

Theorem (Laver)

The set J of nontrivial elementary embeddings on a rank (if nonempty) carries a natural distributive structure, and indeed each $j \in J$ generates a groupoid isomorphic to $\mathbf{F}_{\mathcal{LD}}(1)$.

Laver also exhibited finite quotients $\{\mathbf{LT}_n : n \geq 0\}$ of such free structures, of cardinality 2^n, generalizing

$$\langle \{T, F\}; \Rightarrow \rangle \cong \mathbf{LT}_1$$

Residual Finiteness

Let (L) be the statement "Every two LD-inequivalent terms $t_1(x) \not\equiv_{\mathcal{LD}} t_2(x)$ evaluate differently in some sufficiently large $\boldsymbol{LT}_n$."

Theorem (Laver)

(R2R) $\Rightarrow$ (L)

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Theorem (Laver)

(R2R) $\Rightarrow (L)$

Open Problem

- *(Optimist's version)* (L) *is a theorem of ZFC*
- *(Cautious Optimist's version) Residual finiteness of* $\boldsymbol{F}_{\mathcal{LD}}(1)$ *is a theorem of ZFC*
- *(Pessimist's version)* (L) *is not provable in ZFC alone*
- *(Ultrapessimist's version)* $\neg(L)$ *is a theorem of ZFC*

Slender MLD groupoids

We say $\boldsymbol{G}$ has *Laver dimension n* if

$$\boldsymbol{G} \overset{\pi}{\twoheadrightarrow} \boldsymbol{LT}_n \quad \text{but} \quad \boldsymbol{G} \not\twoheadrightarrow \boldsymbol{LT}_{n+1}$$

and is *slender* if

$$a \equiv_\pi b \quad \Rightarrow \quad \forall x \; a * x = b * x$$

Fact

- *If terms $t_1(x), t_2(x)$ have different right branch length, then there exists a finite zero-dimensional MLD in which they evaluate differently.*
- *If $\boldsymbol{LT}_n \models t_1(x) \approx t_2(x)$ and the terms' right branch lengths are equal, then $\boldsymbol{G} \models t_1(x) \approx t_2(x)$ for every finite slender n-dimensional MLD $\boldsymbol{G}$.*

Isomorphism Classification – Slender Case

Theorem (Many authors, see Drapal 1997)

- *The family $\{\boldsymbol{LT}_n : n \geq 0\}$ is dense in itself and forms a linear inverse system.*
- *The family of (finite) slender MLDs is classified up to isomorphism by n and two function parameters $\rho, \nu : 2^n \to \omega$, which can be chosen independently of each other.*
- *Slender MLDs admit a dense subfamily parametrized by integers $n \geq 0, r \geq 1, v \geq 0$, inverse directed by the usual ordering in n, v and by divisibility in r.*

The existence of a dense subfamily considerably eases the problem of residual finiteness, of course, since if $\boldsymbol{G} \models t_1(x) \not\approx t_2(x)$ and $\boldsymbol{G} \cong \boldsymbol{D}/\theta$ then $\boldsymbol{D} \models t_1(x) \not\approx t_2(x)$ already.

Isomorphism Classification – Nonslender Case

Theorem (Drapal 1997)

The family of all finite MLDs is classified up to isomorphism by n and seven function parameters.

This classification is theoretically nice but of little practical use on its own, since the parameters are highly interdependent. (The full statement of the classification theorem takes about a page.) Virtually every author discussing MLD groupoids restricts most of their attention to the slender case; the nonslender family is known as "combinatorially chaotic".

Main Theorem

Since it isn't a good idea to go sifting through all finite MLDs looking for a disproof of $t_1(x) \equiv_{LD} t_2(x)$, we need better tools.

Theorem (S.)

There exists a family

$$\mathcal{F} = \{\boldsymbol{F}(n, r, v, w_1, w_2) : n \geq 0, r \geq 1, v \geq 2, w_1 \geq 0, w_2 \geq 1\}$$

of finite MLD groupoids, such that

- *Every finite MLD groupoid* $\boldsymbol{G}$ *is a quotient of a member of* $\mathcal{F}$*, and finding one which does so is tractably computable from the multiplication table of* $\boldsymbol{G}$*;*
- $\mathcal{F}$ *is inverse-directed by the usual ordering on* n, v, w_1 *and by divisibility in* r, w_2*.*

Well-behaved?

I refer to the groupoids $\mathcal{F}$ as "well-behaved" for a couple of reasons:

- The five parameters are integers and can be chosen independently of each other.
- $\mathcal{F}$ is inverse-directed, and it is easy to determine whether one member of $\mathcal{F}$ is a quotient of another.
- $\mathcal{F}$ "automatically" separates all terms of different right branch length.

The "combinatorial chaos" in $\mathcal{LD}$ involves basically terms of right branch length 1 and 2. One way of thinking about the groupoids $\mathcal{F}$ is to take a slender groupoid with $v \geq 2$ and split some of its elements, obtained from terms of right branch length 1 or 2, up in pieces in a uniform way.

Room for cautious optimism

Open Problem (ZFC)

Is $\boldsymbol{F}_{\mathcal{LD}}(1)$ residually finite?

Example (Dougherty & Jech)

The function

$$f(m) = \min\{n : \boldsymbol{LT}_n \models 1 * 1 \not\approx 1 * 1_{[2^m+1]}\}$$

if total, grows faster than any primitive recursive function.

However, these terms are clearly not LD-equivalent (they have different right branch lengths).

Room for cautious optimism, cont'd

Example

Let

$$t_1(x) = x_{[5]} * (x_{[2]} * x) \quad t_2(x) = x * ((x * x_{[3]}) * x)$$

We have

$$\boldsymbol{LT}_2 \models t_1 \approx t_2 \quad \text{and } \mathrm{d}_r(t_1) = \mathrm{d}_r(t_2) = 2$$

Hence t_1 and t_2 evaluate identically in every slender MLD groupoid of dimension 2. However we have

$$\boldsymbol{F}(2,3,2,0,1) \models t_1 \not\approx t_2$$

Problems

The groupoids $\mathcal{F}$ provide some level of control or upper bound on the combinatorial explosion present in terms of right branch length ≤ 2.

Open Problem

- *Use $\mathcal{F}$ to improve Dehornoy's normal form result for LD terms in one variable.*
- *Use $\mathcal{F}$ to prove residual finiteness of $\mathbf{F}_{\mathcal{LD}}(1)$.*
- *Inverse limits in $\mathcal{F}$, where at least one of the five parameters is bounded, provide many new examples of infinite nonfree LD groupoids. Do these groupoids represent naturally (e.g. as injection brackets [Dehornoy 2000]) on familiar spaces?*