Independently Axiomatizable $L_{\omega_1,\omega}$ Theories

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Dedication

- This talk is about a paper professor Hjorth and the speaker wrote in 2008, while the speaker was a graduate student.
- The paper *Independently Axiomatizable $L_{\omega_1,\omega}$ Theories* was published in 2009 in the Journal of Symbolic Logic (cf. [1])
- Sadly, professor Hjorth died from heart attack in January 2011.
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Outline

Definition

- A set of sentences T' is called *independent* if for every $\phi \in T'$, $T' \setminus \{\phi\} \not\models \phi$.
- A theory T is called *independently axiomatizable*, if there is a set T' which is independent and T and T' have exactly the same models.

Note: This definition applies to sets of sentences in both first-order ($L_{\omega,\omega}$) and infinitary ($L_{\omega_1,\omega}$) logic, granted that we have defined a meaning for $\models$.

Main Question

When does a theory T have an independent axiomatization?

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The $L_{\omega,\omega}$ Case

Theorem (M.I. Reznikoff- [2])

All theories of any cardinality in $L_{\omega,\omega}$, are independently axiomatizable.

So, for first-order theories the problem is completely resolved. Reznikoff's paper was translated in English (cf. copy on arXiv: [3])

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Theorem (X. Caicedo- [4])

Any $L_{\omega_1,\omega}$- theory of cardinality no more than $\aleph_1$ has an independent axiomatization.

For cardinalities greater than $\aleph_1$, Caicedo obtained partial results for a weaker notion of *countable independence*, which requires that every countable subset of the set of sentences is independent.

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Open Problems

Caicedo also asked whether every $L_{\omega_1,\omega}$- theory has an independent axiomatization or not.

This question appeared on Professor Arnold Miiller's (UW-Madison) website:

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Preliminaries

We work with a *countable* language L.

There are at most $2^{\aleph_0}$ many $L_{\omega_1,\omega}$- sentences.

Under the C.H., $2^{\aleph_0} = \aleph_1$ and problem is solved by Caicedo's theorem.

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1. T is an $L_{\omega_1,\omega}$ theory.
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Scott Analysis

Definition

If $\mathcal{M}$ is a countable model and $\vec{a} \in \mathcal{M}$, define the α-*type* of $\vec{a}$ in $\mathcal{M}$ inductively:

$$\phi_0^{\vec{a},\mathcal{M}} := \bigwedge\{\psi(\vec{x}) | \psi \text{ is atomic or negation of atomic}, \mathcal{M} \models \psi(\vec{a})\},$$

$$\phi_{\alpha+1}^{\vec{a},\mathcal{M}} := \phi_\alpha^{\vec{a},\mathcal{M}} \bigwedge\{\exists \vec{y} \phi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M}\} \wedge$$
$$\bigwedge_n \forall y_0 \dots y_n \bigvee\{\phi_\alpha^{\vec{a}\frown\vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M}\},$$

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If $\mathcal{M}$ is a countable model, then it realizes only countably many types and there is an ordinal $\delta < \omega_1$ such that for all $\vec{a}, \vec{b} \in \mathcal{M}$,

$$\phi_\delta^{\vec{a},\mathcal{M}} = \phi_\delta^{\vec{b},\mathcal{M}} \text{ iff for all } \gamma > \delta, (\phi_\gamma^{\vec{a},\mathcal{M}} = \phi_\gamma^{\vec{b},\mathcal{M}}).$$

The least such ordinal δ we call the Scott height of $\mathcal{M}$ and write $\alpha(\mathcal{M})$. Then $\phi_{\alpha(\mathcal{M})+2}^{\emptyset,\mathcal{M}}$ is called the Scott sentence of $\mathcal{M}$.

Theorem (Scott)

If $\mathcal{N}$ is countable and $\mathcal{N} \models \phi_{\alpha(\mathcal{M})+2}^{\emptyset,\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$.

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Definition

For a $L_{\omega_1,\omega}$-sentence ϕ and $\alpha < \omega_1$, let

$$\Psi_\alpha(\phi) := \{\phi_\alpha^{\vec{a},\mathcal{M}} | \vec{a} \in \mathcal{M}, \mathcal{M} \models \phi\},$$

the α-types of ϕ.
Define also

$$\Phi_\alpha(\phi) := \{\phi_\alpha^{\emptyset,\mathcal{M}} | \mathcal{M} \models \phi\},$$

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Lemma

Let ϕ be a $L_{\omega_1,\omega}$-sentence, $\alpha < \omega_1$, $\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ as defined above and assume that for all $\gamma < \alpha$, $\Psi_\gamma(\phi)$ is countable. Then $\Psi_\alpha(\phi)$ and $\Phi_\alpha(\phi)$ are Σ^1_1 sets.

If $\Psi_\alpha(\phi)$ is as in the above lemma, then by the perfect set theorem for Σ^1_1 sets, it is either countable or has size continuum.

If it is countable, then we can apply the lemma once more and we can keep doing that until we either run out of countable ordinals, or until we find an uncountable $\Psi_{\alpha'}(\phi)$, some $\alpha' > \alpha$.

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If a $L_{\omega_1,\omega}$-sentence ϕ has continuum many non-isomorphic countable models, then there are countable ordinals $\alpha < \beta$, a perfect set P and continuous functions t and M on domain P such that:

- *for all $x \neq y$, $t(x), t(y)$ are distinct α- types of ϕ.*
- *for all x, $M(x)$ is a countable model of ϕ of Scott height $< \beta$,*
- *for all x, $M(x)$ realizes $t(x)$ and*
- *for all $x \neq y$, $M(x) \nvDash t(y)$. In particular, $M(x) \ncong M(y)$.*

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Lemma

The set $A_0 := \{\mathcal{M} | \exists x \in P(\mathcal{M} \cong M(x))\}$ *is Borel.*

Corollary

There is a sentence $\phi^+ \in L_{\omega_1,\omega}$ *such that for every countable model* $\mathcal{M}$,

$$\mathcal{M} \models \phi^+ \text{ iff } \mathcal{M} \in A_0.$$

Lemma

If $\mathcal{N}$ *is a model of* ϕ^+*, countable or uncountable, and it satisfies one of the* $\{t(x) | x \in P\}$*, then it actually satisfies the Scott sentence of* $M(x)$*.*

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The set $A_0 := \{\mathcal{M} | \exists x \in P(\mathcal{M} \cong M(x))\}$ *is Borel.*

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There is a sentence $\phi^+ \in L_{\omega_1,\omega}$ *such that for every countable model* $\mathcal{M}$*,*

$$\mathcal{M} \models \phi^+ \text{ iff } \mathcal{M} \in A_0.$$

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If $\mathcal{N}$ *is a model of* ϕ^+*, countable or uncountable, and it satisfies one of the* $\{t(x) | x \in P\}$*, then it actually satisfies the Scott sentence of* $M(x)$*.*

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Theorem

If ϕ has $2^{\aleph_0}$ many non-isomorphic countable models, then there are P, ϕ^+ and $M(x)$ as above such that

$$\phi \leftrightarrow (\phi \wedge \neg\phi^+) \bigvee_{x \in P} \{s(x) |\ s(x) \text{ is the Scott sentence of } M(x)\}.$$

In particular there exist sentences $\{\phi_\alpha | \alpha \in 2^{\aleph_0}\}$ such that

1. *for all α, ϕ_α is consistent,*
2. $\models \phi \leftrightarrow \bigvee_{\alpha \in I} \phi_\alpha$ *and*
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Note: If sentences $\{\phi_\alpha | \alpha \in I\}$ satisfy properties $(1) - (3)$ above, we say that they *partion* ϕ.

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So, we proved

Theorem

If ϕ has continuum many countable models, then ϕ can be partitioned by continuum many sentences.

Theorem

If there is a sentence $\phi_0 \in T = \{\phi_\alpha | \alpha \in 2^{\aleph_0}\}$ such that $\neg\phi_0$ has continuum many non-isomorphic countable models, then T is independently axiomatizable.

Proof.

We know that there are sentences $\{\psi_\alpha | 0 < \alpha < 2^{\aleph_0}\}$ that partition $\neg\phi_0$.
Define a new theory $T' = \{\overline{\phi_\alpha} | 0 < \alpha < 2^{\aleph_0}\}$ by

$$\overline{\phi_\alpha} : \neg\psi_\alpha \wedge (\neg\phi_0 \vee \phi_\alpha).$$

Then T' is an independent axiomatization of T. □

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More on Scott Analysis

Morley used the Scott analysis to prove the following:

Theorem (Morley)

If ϕ is a $L_{\omega_1,\omega}$- sentence, then ϕ can have

- *either countably many countable models, or*
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For a theory $T = \{\phi_\alpha | \alpha < 2^{\aleph_0}\}$ define

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$$X(T) := \{\mathcal{M} | \mathcal{M} \models \neg\phi, \text{ some } \phi \in T, \ \mathcal{M} \text{ countable}\}$$

Note that all sentences in T_1 provide counterexamples to Vaught's Conjecture.

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Some Results

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In case that $|X(T)| \geq |T_1|$ we will say that T_1 is *small* in T.

Smallness implies that T does not contain too many counterexamples to Vaught's Conjecture.

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Main Theorem

If L is a countable language, T a theory in $L_{\omega_1,\omega}$ and T_1 is small in T (i.e. $|X(T)| \geq |T_1|$), then T is independently axiomatizable.

Corollary

If the Vaught Conjecture holds, then every $T \subset L_{\omega_1,\omega}$ is independently axiomatizable.

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Definition

A collection of Borel sets $\mathcal{B} = \{B_i | i \in I\}$ is independent if

- $\bigcap \mathcal{B} \neq \emptyset$ and
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Two collections $\mathcal{B}, \mathcal{B}'$ are equivalent if $\bigcap \mathcal{B} = \bigcap \mathcal{B}'$.

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Every collection of Borel sets $\mathcal{B} = \{B_i | i \in 2^{\aleph_0}\}$ with $\bigcap \mathcal{B} \neq \emptyset$ admits an equivalent independent collection.

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Open Questions

1. Eliminate the smallness assumption from the main theorem.
2. Prove similar results by replacing $\models$ by $\vdash$.
3. As above by replace $\models$ by $\models_g$, where $T \models_g \phi$ means that in all generic extensions every model of T is also a model of ϕ.
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Boolean Algebras

Let $\phi \leq \psi$ if and only if $\phi \to \psi$. Then the $L_{\omega_1,\omega}$- sentences form a σ-complete Boolean Algebra.

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A set A of sentences is called *σ-filter independent*, if for all ϕ, ϕ is not in the σ-filter generated by $A \setminus \{\phi\}$.

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Very Brief Biography

All the following information/pictures are from the following link:
`http://www.math.ucla.edu/greg.shtml`

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- He died on 13^{th} January 2011 in Melbourne, Australia of a heart attack. He was 47.
- He earned his International Chess Master title in 1984. He played Garry Kasparov among other famous chess players.
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C. R. Acad. Sci., Paris, 260:2385–2388, 1965.

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Xavier Caicedo.
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