

Independently Axiomatizable *L*^ω1,ω Theories ASL 2012 Annual Meeting- Madison, Wisconsin

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April 2*nd*

[Independently Axiomatizable](#page-120-0) $L_{\omega_1,\omega}$ Theories

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- **►** The paper *Independently Axiomatizable L*_{ω1,ω} *Theories* was published in 2009 in the Journal of Symbolic Logic (cf. [\[1\]](#page-117-0))
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Outline

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[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

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- \blacktriangleright A set of sentences T' is called *independent* if for every $\phi \in \mathcal{T}'$, $\mathcal{T}' \setminus {\phi} \neq \phi$.
- \triangleright A theory T is called *independently axiomatizable*, if there is a set T' which is independent and T and T' have exactly the same models.

Note: This definition applies to sets of sentences in both first-order ($L_{\omega,\omega}$) and infinitary ($L_{\omega_1,\omega}$) logic, granted that we have defined a meaning for \models .

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Main Question

When does a theory T have an independent axiomatization?

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The *L*ω,ω Case

Theorem (M.I. Reznikoff- [\[2\]](#page-117-1))

*All theories of any cardinality in L*ω,ω*, are independently axiomatizable.*

So, for first-order theories the problem is completely resolved. Reznikoff's paper was translated in English (cf. copy on arXiv:

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Theorem (X. Caicedo- [\[4\]](#page-117-3))

Any L_{ω1,ω}- theory of cardinality no more than \aleph_1 *has an independent axiomatization.*

For cardinalities greater than \aleph_1 , Caicedo obtained partial results for a weaker notion of *countable independence*, which requires that every countable subset of the set of sentences is independent.

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Preliminaries

We work with a *countable* language *L*.

There are at most 2^{\aleph_0} many $L_{\omega_1,\omega}$ - sentences. Under the C.H., $2^{\aleph_0} = \aleph_1$ and problem is solved by Caicedo's theorem.

So assume that C.H. fails.

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Conventions

For the rest of the talk we assume the following:

- 1. *T* is an $L_{\omega_1,\omega}$ theory.
- 2. When we say that a sentence has "countably many countable models", we mean "countably many non-isomorphic countable models".

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Definition

If M is a countable model and $\vec{a} \in \mathcal{M}$, define the α -type of \vec{a} in M inductively:

 $\phi_0^{\vec{a},\mathcal{M}}$ $\begin{array}{rl} \vec{a}, {\cal M} & := & \bigwedge \{\psi(\vec{X}) | \psi \text{ is atomic or negation of atomic}, {\cal M} \models \psi(\vec{a})\}, \end{array}$ $\begin{array}{lll} \bar{a}_{\cdot} \mathcal{M} & := & \phi_{\alpha}^{\vec{\bm{a}},\mathcal{M}} \bigwedge \lbrace \exists \vec{y} \phi_{\alpha}^{\vec{\bm{a}} \frown \vec{b},\mathcal{M}}(\vec{x},\vec{y}) \vert \vec{b} \in \mathcal{M} \rbrace \wedge \end{array}$ $\bigwedge \forall y_0 \ldots y_n \bigvee \{\phi^{\vec{\bm{a}}}_\alpha \neg \vec{b}, \mathcal{M}(\vec{\bm{\mathsf{x}}}, \vec{\bm{\mathsf{y}}}) | \vec{\bm{b}} \in \mathcal{M} \},$ $\phi_{\lambda}^{\vec{\mathbf{a}},\mathcal{M}}$ $\begin{array}{rcl} \vec{a}, \mathcal{M} & := & \bigwedge \ \phi^{\vec{a},\mathcal{M}}_{\alpha}, \text{ for } \lambda \text{ limit.} \end{array}$

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\phi_{\alpha+1}^{\vec{a},\mathcal{M}} & := & \phi_{\alpha}^{\vec{a},\mathcal{M}} \bigwedge \{ \exists \vec{y} \phi_{\alpha}^{\vec{a} \frown \vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M} \} \wedge \\
& & \bigwedge_{n} \forall y_0 \dots y_n \bigvee \{ \phi_{\alpha}^{\vec{a} \frown \vec{b},\mathcal{M}}(\vec{x},\vec{y}) | \vec{b} \in \mathcal{M} \}, \\
\phi_{\lambda}^{\vec{a},\mathcal{M}} & := & \bigwedge_{\alpha < \lambda} \phi_{\alpha}^{\vec{a},\mathcal{M}}, \text{for } \lambda \text{ limit.}\n\end{array}
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If M is a countable model, then it realizes only countably many types and there is an ordinal $\delta < \omega_1$ such that for all $\vec{\mathbf{a}}, \vec{\mathbf{b}} \in \mathcal{M},$

$$
\phi_\delta^{\vec{\mathbf{a}},\mathcal{M}}=\phi_\delta^{\vec{\mathbf{b}},\mathcal{M}} \text{ iff for all } \gamma>\delta,\, (\phi_\gamma^{\vec{\mathbf{a}},\mathcal{M}}=\phi_\gamma^{\vec{\mathbf{b}},\mathcal{M}}).
$$

The least such ordinal δ we call the Scott height of M and write $\alpha(\mathcal{M}).$ Then $\phi_{\alpha(\mathcal{N})}^{\emptyset, \mathcal{M}}$ $\alpha^{v,\mathcal{M}}_{\alpha(\mathcal{M})+2}$ is called the Scott sentence of $\mathcal{M}.$

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Theorem (Scott)

If N is countable and $\mathcal{N} \models \phi_{\alpha \in \mathcal{N}}^{\emptyset, \mathcal{M}}$ $^{\emptyset, \mathcal{M}}_{\alpha(\mathcal{M})+2}$, then $\mathcal{N} \cong \mathcal{M}$.

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Definition

For a $L_{\omega_1,\omega}$ -sentence ϕ and $\alpha < \omega_1$, let

$$
\Psi_{\alpha}(\phi) := \{ \phi_{\alpha}^{\vec{\mathbf{a}},\mathcal{M}} | \vec{\mathbf{a}} \in \mathcal{M}, \mathcal{M} \models \phi \},
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the α -types of ϕ . Define also

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If $\Psi_{\alpha}(\phi)$ is as in the above lemma, then by the perfect set theorem for Σ_1^1 sets, it is either countable or has size continuum.

If it is countable, then we can apply the lemma once more and we can keep doing that until we either run out of countable ordinals, or until we find an uncountable $\Psi_{\alpha'}(\phi)$, some $\alpha'>\alpha.$

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*If a L*ω1,ω*-sentence* φ *has continuum many non-isomorphic countable models, then there are countable ordinals* α < β*, a perfect set P and continuous functions t and M on domain P such that:*

- **F** for all $x \neq y$, $t(x)$, $t(y)$ are distinct α -types of ϕ .
- \triangleright for all x, $M(x)$ *is a countable model of* ϕ *of Scott height*
- \triangleright *for all x, M(x) realizes t(x) and*
- \triangleright *for all x* \neq *y, M(x)* \neq *t(y). In particular, M(x)* \cong *M(y).*

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*The set A*₀ := { \mathcal{M} $\exists x \in P(\mathcal{M} \cong M(x))$ } *is Borel.*

Corollary

There is a sentence $\phi^+ \in L_{\omega_1,\omega}$ *such that for every countable model* M*,*

 $\mathcal{M} \models \phi^+$ iff $\mathcal{M} \in A_0$.

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Lemma

If N *is a model of* ϕ^+ , countable or uncountable, and it satisfies *one of the* $\{t(x)|x \in P\}$ *, then it actually satisfies the Scott sentence of M*(*x*)*.*

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Lemma

There exists an L_{ω1,ω}-sentence that expresses the fact that a model satisfies one of the types in $\{t(x)|x \in P\}$ $\{t(x)|x \in P\}$ *[.](#page-50-0)*

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If ϕ has 2^{No} many non-isomorphic countable models, then there *are P*, ϕ^+ *and M(x) as above such that*

 $\phi \leftrightarrow (\phi \wedge \neg \phi^+) \bigvee \{s(x) | s(x) \text{ is the Scott sentence of } M(x)\}.$ *x*∈*P*

In particular there exist sentences $\{\phi_\alpha | \alpha \in 2^{\aleph_0}\}$ *such that*

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Note: If sentences $\{\phi_{\alpha} | \alpha \in I\}$ satisfy properties (1) – (3) above, we say that they *partion* ϕ .

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So, we proved

Theorem

If φ *has continuum many countable models, then* φ *can be partitioned by continuum many sentences.*

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If there is a sentence $\phi_0 \in \mathcal{T} = \{ \phi_\alpha | \alpha \in 2^{\aleph_0} \}$ *such that* $\neg \phi_0$ *has continuum many non-isomorphic countable models, then T is independently axiomatizable.*

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\overline{\phi_{\alpha}}: \neg \psi_{\alpha} \wedge (\neg \phi_0 \vee \phi_{\alpha}).
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Proof.

We know that there are sentences $\{\psi_{\alpha}|0 < \alpha < 2^{\aleph_0}\}$ that *partition* $\neg \phi_0$.

Define a new theory $T' = \{ \overline{\phi_\alpha} | 0 < \alpha < 2^{\aleph_0} \}$ *by*

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Theorem (Morley)

If ϕ *is a* $L_{\omega_1,\omega}$ - sentence, then ϕ can have

- **E** either countably many countable models, or
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X(T) := \{ \mathcal{M} | \mathcal{M} \models \neg \phi \text{, some } \phi \in T, \mathcal{M} \text{ countable} \}
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Note that all sentences in T_1 provide counterexamples to Vaught's Conjecture.

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[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

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[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

Definition In case that $|X(T)| \geq |T_1|$ we will say that T_1 is *small* in T.

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[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

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Main Theorem

If L is a countable language, T a theory in $L_{\omega_1,\omega}$ *and T₁ <i>is small in T (i.e.* $|X(T)| \ge |T_1|$ *), then T is independently axiomatizable.*

Corollary

If the Vaught Conjecture holds, then every T ⊂ *L*_{ω1,ω} *is independently axiomatizable.*

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A collection of Borel sets $\mathcal{B} = \{\textit{B}_{\textit{i}} | \textit{i} \in \textit{I}\}$ is independent if

$\blacktriangleright \bigcap \mathcal{B} \neq \emptyset$ and

▶ for every $i \in I$, $\bigcap_{j \neq i} B_j \setminus B_i \neq \emptyset$

Two collections $\mathcal{B}, \mathcal{B}'$ are equivalent if $\bigcap \mathcal{B} = \bigcap \mathcal{B}'.$

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- ▶ for every $i \in I$, $\bigcap_{j \neq i} B_j \setminus B_i \neq \emptyset$

Two collections $\mathcal{B}, \mathcal{B}'$ are equivalent if $\bigcap \mathcal{B} = \bigcap \mathcal{B}'.$

Theorem

 $\bm{\mathit{Every}}$ collection of Borel sets $\mathcal{B} = \{\bm{B}_i | i \in 2^{\aleph_0}\}$ with $\bigcap \mathcal{B} \neq \emptyset$ *admits an equivalent independent collection.*

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- 1. Eliminate the smallness assumption from the main theorem.
- 2. Prove similar results by replacing \models by \vdash .
- 3. As above by replace \models by \models _{*a*}, where $T \models$ _{*a*} ϕ means that in all generic extensions every model of *T* is also a model of ϕ .
- 4. Prove that any $L_{\omega_1,\omega}$ theory is independently axiomatizable, even for uncountable languages. Our techniques here rely heavily on this assumption.

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So, given a set of sentences A to find another set A' such that

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All the following information/pictures are from the following link: <http://www.math.ucla.edu/greg.shtml>

[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

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- ► He died on 13th January 2011 in Melbourne, Australia of a heart attack. He was 47.
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[Independently Axiomatizable](#page-0-0) $L_{\omega_1,\omega}$ Theories

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- Greg Hjorth and Ioannis A. Souldatos. 歸 Independently axiomatizable $\mathcal{L}_{\omega_{\infty},\omega}$ theories. *J. Symb. Log.*, 74(4):1273–1286, 2009.
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