On Peterzil – Steinhorn groups definable in algebraically closed fields.

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Peterzil – Steinhorn Theorem

In the paper *Definable compactness and definable subgroups of o-minimal groups (1999)* K. Peterzil and C. Steinhorn showed that to any “unbounded” curve in an o-minimal group one can associate a one-dimensional “non-compact” subgroup.

Peterzil – Steinhorn Theorem

Theorem (Peterzil–Steinhorn)

Let G be a group definable in an o-minimal structure. Let $\sigma\colon (a,b) \to G$ be a curve such that the limit $\lim\limits_{t\to b^-} \sigma(t)$ does not exist in G. Then the set of all limits

$$H = \operatorname*{Lim}_{t_1\to b,\ t_2\to b} \sigma(t_1)\cdot\sigma(t_2)^{-1}$$

is a one dimensional "non-compact" subgroup of G.

We will denote the above subgroup H by $PS[\sigma]$ and call it *(left) Peterzil–Steinhorn subgroup of σ in G.*

Remark

We can also define the right Peterzil–Steinhorn subgroup as the set of all limits

$$H_r = \operatorname*{Lim}_{t_1\to b,\ t_2\to b} \sigma(t_1)^{-1}\cdot\sigma(t_2)$$

Left vs. Right

Let $\sigma\colon (0,\infty) \to G$ be a definable curve. If

$$g \in PS[\sigma] = \operatorname*{Lim}_{t_1\to\infty,\ t_2\to\infty} \sigma(t_1)\cdot\sigma(t_2)^{-1}$$

Then writing $g \sim \sigma(\infty)\sigma(\infty)^{-1}$ we have $g\cdot\sigma(\infty) \sim \sigma(\infty)$, and $PS[\sigma]$ can be viewed as "the left stabilizer" of $\sigma(\infty)$.

For the same reason the right PS-subgroup

$$\operatorname*{Lim}_{t_1\to\infty,\ t_2\to\infty} \sigma(t_1)^{-1}\cdot\sigma(t_2)$$

can be viewed as "the right stabilizer" of $\sigma(\infty)$:

$$\{g \in G\colon \sigma(\infty) \sim \sigma(\infty)g\}.$$

Some Examples

Example

Let $\sigma(t)\colon (a,b) \to G$ be a continuous curve. If the image of σ in G is a subgroup H of G then $PS[\sigma] = H$.

Example

Let $\sigma\colon (0,\infty) \to (\mathbb{R},+)^2$ be an unbounded curve. After reparametrization we may assume $\sigma(t) = (t, y(t))$. Then $PS[\sigma]$ is the line through the origin with the slope

$$a = \lim_{t\to\infty} \frac{d}{dt} y(t).$$

Remark

In the above example PS-subgroup is just the usual linear asymptote of σ at infinity.

Some Examples

Example

Let $\sigma\colon (0,\infty) \to (\mathbb{R}^{>0},\cdot)^2$ be a semi-algebraic curve such that $\lim_{t\to\infty} \sigma(t)$ does not exist. After reparametrization we may assume $\sigma(t) = (t^k, y(t))$. Write $y(t) = at^q + o(t^q)$ with $a \neq 0 \in \mathbb{R}$. Then

$$PS[\sigma] = \{(t^k, t^q)\colon t > 0\}.$$

Remark

In general, for a curve $\sigma\colon (0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ it is not easy to detect what $PS[\sigma]$ is.

Exercise

Compute $PS[\sigma]$ for

$$\sigma(t) = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}.$$

PS-Subgroups Over $\mathbb{C}$

Let $C \subseteq \mathrm{GL}(n, \mathbb{C})$ be a complex algebraic curve.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $\mathbb{C} = \mathbb{R} \oplus i\,\mathbb{R}$ we can view $\mathrm{GL}(n, \mathbb{C})$ as a subgroup G of $\mathrm{GL}(2n, \mathbb{R})$.

Under this identification C is a semi-algebraic set of $\mathbb{R}$-dimension 2, and it is unbounded. Let $\sigma\colon (0, \infty) \to C$ be an unbounded semi-algebraic curve.

Working in $\mathbb{R}$ we obtain a semi-algebraic subgroup $PS[\sigma]$ of G of $\mathbb{R}$-dimension one.

Let H be the Zariski closure of $PS[\sigma]$ in $\mathrm{GL}(n, \mathbb{C})$. It is a complex-algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$ of complex dimension one.

Thus to every algebraic curve C in $\mathrm{GL}(n, \mathbb{C})$ we can assign a one–dimensional algebraic PS–subgroup!

Hilbert – Mumford Criterion

Theorem (Hilbert – Mumford)

Let $G < \mathrm{GL}(n, \mathbb{C})$ be a reductive algebraic group, and $\vec{a} \in \mathbb{C}^n$. Assume $\vec{0} \in \mathrm{cl}(G \cdot \vec{a})$. Then there is a one–parameter subgroup $H < G$ such that $\vec{0} \in cl(H \cdot \vec{a})$. (One–parameter: there is an algebraic group isonorphism $\varphi \colon \mathbb{C}^* \to H$.)

This theorem is a key in constructing algebraic quotients $G \setminus \mathbb{C}^n$.

Some Questions

Question 1

Let $C \subseteq \mathrm{GL}(n, \mathbb{C})$ be a complex algebraic curve.
To get a PS-subgroup associated with C we identified $\mathbb{C}$ with $\mathbb{R}^2$ and used $\mathbb{R}$-topology.

But there are infinitely many real closed fields R with $\mathbb{C} = R \oplus iR$, and we could use another R-semialgebraic structure on $\mathbb{C}$.

Do we always get the same PS-subgroups?

Question 2

If PS-subgroups over $\mathbb{C}$ do not depend on the choice of a real closed subfield, can we constructed them“algebraically”? Can we do it in all characteristics?

PS-Subgroups Redefined

Let $\sigma\colon (0,\infty) \to GL(n,\mathbb{R})$ be an unbounded semialgebraic curve. Recall that

$$PS[\sigma] = \lim_{t_1 \to b^-,\ t_2 \to b^-} \sigma(t_1)\cdot\sigma(t_2)^{-1}.$$

Let $\mathcal{R} \succ \mathbb{R}$ be a proper elementary extension, and let $\mathcal{O} \subset \mathcal{R}$ be the convex hull of $\mathbb{R}$.

We can write $\mathcal{O}$ as the disjoint union $\mathcal{O} = \dot{\bigcup}\{r+\mu\colon r\in\mathbb{R}\}$, where μ is the set of infinitesimally small elements. We have the standard part mapping $\mathrm{st}\colon \mathcal{O} \to \mathbb{R}$ defined by $\mathrm{st}(r+\mu) = r$ for $r \in \mathbb{R}$.

Let $\tau \in \mathcal{R}\setminus\mathbb{R}$ be a large positive nonstandard element. Let $\sigma(\mathcal{R}) \subseteq \mathrm{GL}(n,\mathcal{R})$ be the image of $(0,\infty) \subseteq \mathcal{R}$ under σ.

Claim

Viewing $\mathrm{GL}(n,\mathcal{R})$ as a subset of $\mathcal{R}^{n^2}$ we have

$$PS[\sigma] = \mathrm{st}\left(\left[\sigma(\mathcal{R})\cdot\sigma(\tau)^{-1}\right] \bigcap \mathcal{O}^{n^2}\right)$$

Summary.

$$PS[\sigma] = \mathrm{st}\left(\left[\sigma(\mathcal{R}) \cdot \sigma(\tau)^{-1}\right] \bigcap \mathcal{O}^{n^2}\right)$$

To get a PS-subgroup for an algebraic curve $C \subseteq \mathrm{GL}(n, k)$ we need:

- A "branch" of C at infinity.
- A "standard part" mapping.

Algebraic Preliminaries

Let k be an algebraically closed field.

Let $C \subseteq \mathrm{GL}(n,k)$ be an irreducible algebraic curve. We view $\mathrm{GL}(n,k)$ as a subset of k^m with $m = n^2$.

As usual:

- $I_C \subset k[x_1, \ldots, x_m]$ is the ideal of polynomial vanishing on C;
- $k[C] = k[\bar{x}]/I_c$ is the ring of regular functions on C;
- $k(C)$ is the field of rational functions on C (It is the field of fractions of $k[C]$).

Let $\bar{C}$ be the Zariski closure of C in $\mathbb{P}^m(k)$. We assume $\bar{C}$ is smooth. The set $\bar{C} \setminus C$ is finite, and for $\rho \in \bar{C} \setminus C$ let

$$\Sigma_\rho(\bar{x}) = \{r(\bar{x}) \in k(C) \colon r(\rho) = 0\}.$$

Remark

Since $\bar{C}$ is smooth, the set Σ_ρ determines all values $r(\rho)$, $\rho \in k(C)$.

Getting a standard part mapping

Let $L > k$ be a proper algebraically close extension of k.

We choose a valuation ring $\mathcal{O} \subset L$ containing k such that the residue field of $\mathcal{O}$ is k.

In other words, we choose a subring $k \subset \mathcal{O} \subset L$ such that

- $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$ for any $a \neq 0 \in L$;
- there is a ring homomorphsim $\mathrm{st}\colon \mathcal{O} \to k$ such that $\mathrm{st} \restriction k = \mathrm{id}_k$.

For $\mu = \mathrm{st}^{-1}(0)$ we have that $\mathcal{O}$ is the disjoint union $\mathcal{O} = \dot{\bigcup}\{a + \mu \colon a \in k\}$ with $\mathrm{st}(a + \mu) = a$ for $a \in k$.

For $x, y \in L$ with $x \neq 0$ we define $v(x) \leqslant v(y) \iff x^{-1}y \in \mathcal{O}$.

Basic Facts

Let $\mathcal{L}_v$ be the language of rings $(+, \cdot, -, 0, 1)$ augmented by a binary relational symbol.
We consider L as an $\mathcal{L}_v$-structure by interpreting the binary relation as $v(x) \leqslant v(y)$.

It is not hard to see that both $\mathcal{O}$ and μ are $\mathcal{L}_v$-definable:

$$\mathcal{O} = \{y \in L \colon v(1) \leqslant v(y)\}, \quad \mu = \{x \in L \colon \neg v(x) \leqslant v(1)\}.$$

Fact

1. *L has a quantifier elimination in the language $\mathcal{L}_v$.*
2. *Let $X \subseteq L^m$ be a $\mathcal{L}_v$-definable subset (with parameters from L). Then the image in k^m of the set $X \cap \mathcal{O}^m$ ander the map st is definable in the language of rings.*
   Moreover if X is algebraic then $\mathrm{st}(X \cap \mathcal{O}^m)$ has dimension at most of X.

A branch at infinity

We have $C \subseteq \mathrm{GL}(n, k)$.

We fix $\rho \in \bar{C} \setminus C$. Let $\Sigma(\bar{x}) = \Sigma_\rho(\bar{x}) \subseteq k(C)$.

As usual we denote by $C(L)$ the set of L-points on C.

Let $C^\infty = C(L) \setminus \mathcal{O}^m$ and $C^\infty_\rho = \{x \in C^\infty \colon r(x) \in \mu \text{ for all } r \in \Sigma(\bar{x})\}$.

Claim

The set C^∞_ρ is $\mathcal{L}_v$-definable over k.

Proof.

Follows from the existence of a uniformizing parameter. □

Recall $C_\rho^\infty = \{x \in C^\infty \colon r(x) \in \mu$ for all $r \in \Sigma\}$.

Claim

Every two elements $\alpha, \beta \in C_\rho^\infty$ have the same type over k (in the language $\mathcal{L}_v$).

Proof.

By quantifier elimination we need to show that for any $p(\bar{x}), q(\bar{x}) \in k[\bar{x}]$

$$v(p(\alpha)) \not\leq v(q(\alpha)) \text{ iff } v(p(\beta)) \not\leq v(p(\beta)).$$

It is not hard to see that

$$v(p(\alpha)) \not\leq v(q(\alpha)) \text{ iff } v(p(\alpha)/q(\alpha)) \in \mu \text{ iff } p(\bar{x})/q(\bar{x}) \in \Sigma(\bar{x}).$$

□

PS-subgroup

We fix $\beta \in C_\rho^\infty$.
Let $H \subseteq \mathrm{GL}(n,k)$ be the image of $C_\rho^\infty \cdot \beta^{-1} \cap \mathcal{O}^m$ under the map st.

Claim

H is an algebraic subgroup of $\mathrm{GL}(n,k)$.

Proof.

We show that H is closed under multiplication.
Assume $h_1, h_2 \in H$. We need to show that $h_1 \cdot h_2$ is in H.
Let $\alpha_1, \alpha_2 \in C_\rho^\infty$ be such that $\alpha_i \beta^{-1} \in h_i + \mu^m$ for $i = 1, 2$.
Since α_2 and β realize the same type over k there is $\alpha_1' \in C_\rho^\infty$ with $\alpha_1' \cdot \alpha_2^{-1} \in h_1 + \mu^m$. Hence

$$\alpha_1' \cdot \beta^{-1} = (\alpha_1' \cdot \alpha_2^{-1}) \cdot (\alpha_2 \cdot \beta^{-1}) \in (h_1 + \mu^m) \cdot (h_2 + \mu^m).$$

Since the group operations are defined by polynomial maps over k we have $(h_1 + \mu^m) \cdot (h_2 + \mu^m) \subseteq (h_1 \cdot h_2) + \mu^m$ and $h_1 \cdot h_2 \in H$. □

PS-Subgroup

Claim

$H = \mathrm{st}\left(C_\rho^\infty \cdot \beta^{-1} \cap \mathcal{O}^m\right)$ *is a one-dimensional subgroup of* $\mathrm{GL}(n, k)$

Proof.

We only need to show that it is infinite.
Assume H is finite. Then $C_\rho^\infty \cdot \beta^{-1}$ would be covered by finitely many disjoint open balls $a_i + \mu^m$, and the curve $C(L) \cdot \beta^{-1}$ would be covered nontrivially by a finite disjoint union of open balls.
By a result of Hrushovski and Loeser, every irreducible curve in L is $v + g$-connected. A contradiction. □

Remark

The subgroup H does not depend on the choice of β and L. But it may depend on the choice of the point ρ in $\bar{C} \setminus C$.
We will denote this subgroup by $PS[C_\rho]$.

Algebraic Definition?

Question: Is it possible to define $PS[C_\rho]$ working enirely in k?

Conjecture. $PS[C_\rho]$ is "the left stabilizer" of ρ.

By *a left compactification* of $\mathrm{GL}(n,k)$ we mean a complete variety V with an embedding $GL(n,k) \hookrightarrow V$ so that the action of $\mathrm{GL}(n,k)$ on itself by multiplication on the left extends to an action on V.

Claim

Let $C \subset GL(n,k)$ be a curve, and $GL(n,k) \hookrightarrow V$ be a left compactification. Let $\bar{C} \subset V$ be the Zariski closure of C in V and $\rho \in \bar{C} \setminus C$. Then

$$PS[C_\rho] \subseteq Stab(\rho) = \{g \in \mathrm{GL}(n,k) \colon g \cdot \rho = \rho\}.$$

Question

Let $C \subset GL(n,k)$ be a curve. Is there a left compactification $GL(n,k) \hookrightarrow V$ such that for any $\rho \in \bar{C} \setminus S$ we have $PS[C_\rho] = Stab(\rho)$?

A Problem

It fails for projective compactifications.

Let $\xi\colon \mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(N,\mathbb{C}) \subseteq \mathbb{C}^{N\times N}$ be an embedding, and $\pi\colon \mathbb{C}^{N\times N} \to \mathbb{P}^{(N\times N)-1}(\mathbb{C})$ be the projection.

The Zariski closure $[GL(n,\mathbb{C})]_\xi$ of $\pi\circ\xi(\mathrm{GL}(n,\mathbb{C}))$ is called *a projective compactification* of $\mathrm{GL}(n,\mathbb{C})$.

Example

Let C be the Zariski closure of

$$\sigma(t) = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$$

in $\mathrm{GL}(2,\mathbb{C})$. Its PS-subgroup is isomorphic to $(\mathbb{C},+)$.
But, due to Hilbert–Mumford criterion, for any projective compactification $[\mathrm{GL}(2,\mathbb{C})]_\xi$ and a point $\rho \in \bar{C} \setminus C$ the stabilizer $Stab(\rho)$ contains a one–parameter subgroup.
In particular the dimension $Stab(\rho)$ is at least 2.

Is it addition or multiplication?

Up-to a definable isomorphism there are exactly two non-compact groups definable in the field of reals: $(\mathbb{R}, +)$ and $(\mathbb{R}^{>0}, \cdot)$.

Question

Let $\sigma\colon (0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ be an unbounded semialgebraic curve. How to detect if $PS[\sigma]$ is additive or multiplicative?

Example

For

$$\sigma(t) = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$$

the Peterizl-Steinhorn subgroup is additive.

Is it addition or multiplication?

Remark

In general the growth rate of $\sigma(t)$ does not provide enough information to detect the nature of $PS[\sigma]$.

There is an unbounded semialgebraic curve in $\mathrm{GL}(2,\mathbb{R})$ whose (left) PS-subgroup is additive but the right PS-subgroup is multiplicative.

Conjecture [G. Poulios]

Let $\sigma\colon (0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ be an unbounded semi-algebraic curve. Let $\lambda \in \mathbb{R}$ be such that

$$\lim_{t\to\infty} t^{\lambda}\, \dot{\sigma}(t)\, \sigma(t)^{-1}$$

exists (in the space of all $(n \times n)$ matrices) and is nonzero. Then

- $PS[\sigma]$ is additive if and only if $\lambda < 1$;
- $PS[\sigma]$ is multiplicative if and only if $\lambda = 1$;