The hereditarily ordinal definable sets in models of determinacy

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March 2012

Plan:

I. Absolute fragments of HOD.
II. Some results on HOD^M, for $M \models \mathrm{AD}$.
III. Mice, and their iteration strategies.
IV. HOD^M as a mouse.

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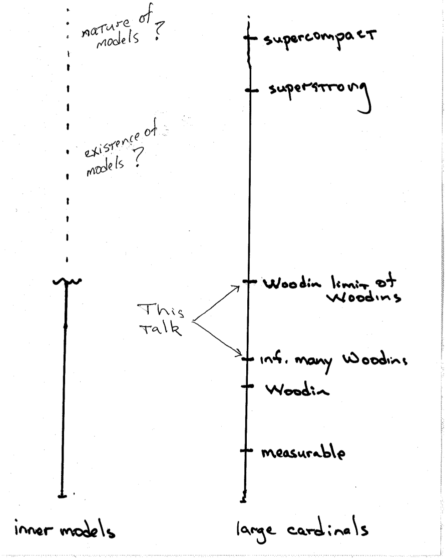
Theorem (Gödel, late 30s?)

Assume ZF*; then* $HOD \models \mathrm{ZFC}$.



nature of models ?
existence of models ?
supercompact
superstrong
Woodin limit of Woodins
This talk
inf. many Woodins
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(2) Large cardinals *do* decide the theory of $L(\mathbb{R})$, and hence that of $\mathrm{HOD}^{L(\mathbb{R})}$.

(3) In fact, they decide the theory of $L(\Gamma, \mathbb{R}))$, for boldface pointclasses $\Gamma \subsetneq P(\mathbb{R})$ of "well-behaved" sets of reals.

(A *real* is an infinite sequence of natural numbers. A *boldface pointclass* is a collection of sets of reals closed under complements and continuous pre-images.)

Homogeneously Suslin sets of reals

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function $x \mapsto \langle (M^x_n, i^x_{n,m}) \mid n, m < \omega \rangle$ on ω^ω such that for all x, $M^x_0 = V$, each M^x_n is closed under κ-sequences, and

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Theorem (Martin, S., Woodin)

If there are arbitrarily large Woodin cardinals, then for any pointclass Γ properly contained in Hom_∞, every set of reals in $L(\Gamma, \mathbb{R})$ is in Hom_∞, and thus $L(\Gamma, \mathbb{R}) \models$ AD.

Generic absoluteness

A $(\Sigma^2_1)^{Hom_\infty}$ *statement* is one of the form:
$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \varphi$.

Theorem (Woodin)

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Woodin's Ω-conjecture says that, granting there are arbitrarily large Woodin cardinals, all generic absoluteness comes via reductions to $(\Sigma_1^2)^{Hom_\infty}$ statements.

Open questions: Does any large cardinal hypothesis (e.g. the existence of arbitarily large supercompact cardinals) imply

(1) that statements of the form $\forall x \in \mathbb{R} \exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \varphi[x]$ are absolute for set forcing?

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The canonical inner models for such a large cardinal hypothesis would have to be different in basic ways from those we know.

It is unlikely that superstrong cardinals would suffice.

Conjecture. Assume there are arbitrarily large Woodin cardinals, and let $\Gamma \subsetneq P(\mathbb{R})$ be a pointclass; then $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathrm{GCH}$.

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Such a theory has been developed for M below the minimal model of $\mathrm{AD}_\mathbb{R} +$"Θ is regular."

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Definition (Suslin representations)

Let $A \subseteq \mathbb{R}$ and $\kappa \in \mathrm{OR}$; then A is κ-*Suslin* iff there is a tree T on $\omega \times \kappa$ such that $A = p[T] = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T\}$.

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(c) *Suppose* $\exists A \subseteq \mathbb{R}(V_{\omega+1}, \in, A) \models \varphi$; *then* $HOD \models (\exists A \subseteq \mathbb{R}(V_{\omega+1}, \in, A) \models \varphi)$.

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So HOD^M can see a surrogate for M.

The Solovay sequence

Definition

(AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set

$$\begin{aligned}
\theta_0 &= \theta(\emptyset), \\
\theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\
\theta_\lambda &= \bigcup_{\alpha<\lambda} \theta_\alpha.
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$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$. Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B.

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$L(\mathbb{R}) \models \theta_0 = \Theta$.

Theorem (Woodin, mid 80's)

Assume AD^+*, and suppose* A *and* $\mathbb{R} \setminus A$ *are Suslin; then*

(a) *All* $\Sigma^2_1(A)$ *sets of reals are Suslin, and*

(b) *All* $\Pi^2_1(A)$ *sets are Suslin iff all* $OD(A)$ *sets are Suslin iff* $\theta(A) < \Theta$.

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Theorem (Martin, Woodin, mid 80's)

Assume AD^+*; then the following are equivalent:*

(1) $\mathsf{AD}_{\mathbb{R}}$,

(2) *Every set of reals is Suslin,*

(3) $\Theta = \theta_\lambda$*, for some limit* λ.

Theorem (Woodin late 90s, S. 2007)

The following are equiconsistent:

(1) $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$,

(2) $\mathsf{ZFC} + \exists\lambda(\lambda$ *is a limit of Woodins and* $< \lambda$*-strong cardinals).*

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Remark. The consistency strengths of the following have been precisely calibrated:

(1) $\mathsf{ZF} + \mathsf{AD}^+ + \theta_\omega = \Theta$

(2) $\mathsf{ZF} + \mathsf{AD}^+ + \theta_{\omega_1} = \Theta$ (Woodin late 90s, S. 2007),

(3) $\mathsf{ZF} + \mathsf{AD}^+ + \theta_{\omega_1} < \Theta$ (Sargsyan, S. 2008),

(4) $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular (Sargsyan 2009, Sargsyan-Zhu 2011).

All are weaker than a Woodin limit of Woodin cardinals. The arguments use the theory of HOD^M, for $M \models \mathsf{AD}^+$.

Large cardinals in HOD

Theorem

Assume AD*; then*

(a) Θ *is a limit of measurable cardinals (Solovay, Moschovakis, late 60's).*

(b) *Every measure on a cardinal* $< \Theta$ *is ordinal definable (Kunen, early 70's).*

(c) *HOD* $\models \Theta$ *is a limit of measurable cardinals.*

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Assume AD*, ; then*

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Key Question: Can there be any other Woodin cardinals in HOD?

Mice, and their iteration strategies

More was proved about HOD^M, for $M \models \mathrm{AD}^+$, using the tools of descriptive set theory. E.g. Becker proved various instances of GCH, and that ω_1^V is its least measurable cardinal.

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Coherence: for all $\alpha \leq \gamma$, $E_\alpha = \emptyset$, or E_α is an extender (system of ultrafilters) with support α over $\mathcal{M}|\alpha = (J_\alpha^{\vec{E}\restriction\alpha}, \in, \vec{E} \restriction \alpha)$ coding

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such that

$$i(\vec{E} \restriction \alpha) \restriction \alpha = \vec{E} \restriction \alpha \text{ and } i(\vec{E} \restriction \alpha)_\alpha = \emptyset.$$

Remark. The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

Proper class premice are sometimes called extender models.

A *mouse* is an iterable premouse.

The iteration game

Let $\mathcal{M}$ be a premouse. In $\mathcal{G}(\mathcal{M}, \theta)$, players I and II play for θ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$ at the base.

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Round $\beta + 1$*:* I picks an extender E_β from the sequence of $\mathcal{M}_\beta$, and $\xi \leq \beta$. We set

$$\mathcal{M}_{\beta+1} = \mathrm{Ult}(\mathcal{M}_\xi, E_\beta),$$

I must choose ξ so that this ultrapower makes sense.

Round λ*, for* λ *limit:* II picks a branch b of $\mathcal{T}$ which is cofinal in λ, and we set

$$\mathcal{M}_\lambda = \text{ dirlim }_{\alpha \in b} \mathcal{M}_\alpha.$$

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Round λ*, for* λ *limit:* II picks a branch b of $\mathcal{T}$ which is cofinal in λ, and we set

$$\mathcal{M}_\lambda = \text{ dirlim }_{\alpha \in b} \mathcal{M}_\alpha .$$

As soon as an illfounded model $\mathcal{M}_\alpha$ arises, player I wins. If this has not happened after θ rounds, then II wins.

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Corollary

If $\mathcal{M}$ is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap \mathcal{M}$, then x is ordinal definable.

Constructing $\omega_1 + 1$-iterable countable mice is the central problem of inner model theory. The way to do it is to construct an absolutely definable (i.e. Hom_∞) ω_1-strategy.

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(AD^+) *Mouse Capturing* (MC) is the statement: for any reals x, y, the following are equivalent:

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Mouse Set Conjecture: Assume AD^+, and that there is no ω_1-iteration strategy for a mouse with a superstrong cardinal; then Mouse Capturing holds.

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$$\exists A (V_{\omega+1}, \in, A) \models \varphi[x]$$

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$$M \models \text{ ZC + "there are arbitrarily large Woodin cardinals"},$$

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That is, Σ^2_1 truth is captured by mice.

HOD^M as a mouse

Theorem (Woodin, S. early 90s)

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(1) $HOD^{L(\mathbb{R})}$ *is a premouse up to* $\Theta^{L(\mathbb{R})}$,

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What is the full $\mathrm{HOD}^{L(\mathbb{R})}$? A new species of mouse!

Let M_ω be the canonical minimal extender model with ω Woodins, and Σ its *unique* iteration strategy. Then

$$\mathrm{HOD}^{L(\mathbb{R})} = L[N, \Lambda],$$

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(Woodin, 1995.) The iteration strategy Λ is new canonical information. (No iterable extender model with a Woodin knows how to iterate itself for iteration trees based on its bottom Woodin.) Nevertheless, Λ adds no new bounded subsets of Θ beyond those already in N, and it preserves the Woodinness of Θ.

HOD-mice

Work of Woodin (late 90s) and Sargsyan (2008) led to an analysis of HOD^M as a *hod-mouse*, for $M \models \mathrm{AD}^+$ up to the minimal model of $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular. In such M:

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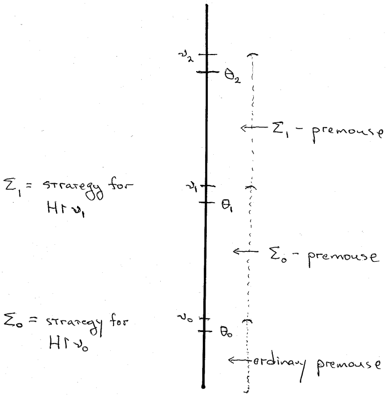

$$H = \mathrm{HOD}^M$$

for $M \models AD^+$

The core model induction method

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Let Γ be the pointclass of currently captured sets (via mice with iteration strategies in Γ. We have $\Gamma \models \mathsf{AD}^+$.

Now we use

(1) Our strong hypothesis,
(2) core model theory (covering theorem, etc.), and
(3) the descriptive set theory of $L(\Gamma, \mathbb{R})$, esp. the analysis of its HOD,

to construct mice capturing more sets of reals.

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$Con(\mathrm{ZFC} + \mathrm{PFA})$ implies $Con(\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular).

Holy Grail: Con(ZFC + PFA) implies Con(ZFC+ "there is a supercompact cardinal").

Beyond $AD_{\mathbb{R}} + \Theta$ regular

Definition

LST is the theory: $\mathsf{ZF} + \mathsf{AD}^+ +$ "$\Theta = \theta_{\lambda+1}$, where θ_λ is the largest Suslin cardinal."

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LST is the theory: ZF + AD^+ + "$\Theta = \theta_{\lambda+1}$, where θ_λ is the largest Suslin cardinal."

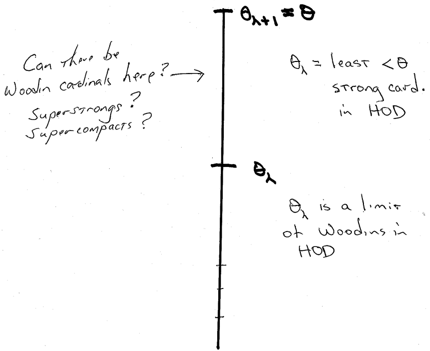
LST implies that for $\Gamma = \{A \mid w(A) < \theta_\lambda\}$, $L(\Gamma, \mathbb{R}) \models \Theta$ is regular.

Probably:

Theorem (Sargsyan, S. 2009–)

If M is the minimal model of LST*, then $HOD^M \models$* GCH*.*

Probably, one can construct a model of LST from a little more than a Woodin limit of Woodins, but this is open now.



HOD^M, for $M \models \mathrm{LST}$

Key Question: In the LST situation, can HOD have Woodin cardinals strictly between the largest Suslin cardinal and Θ? Can it have superstrongs, or supercompacts, or... in that interval? If so:

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(1) The comparison problem for hod mice becomes much harder.
(2) A *Vision of ultimate K* becomes possible.

Is V a hod mouse?

The following is an axiom recently proposed by Hugh Woodin:

- if
$$\exists \alpha (V_\alpha \models \varphi),$$
then for some $M \models \mathrm{AD}^+$ such that $\mathbb{R} \cup \mathrm{OR} \subseteq M$,
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(c) It may be consistent with all the large cardinal hypotheses.