The Ginsburg–Sands theorem and computability theory

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16 May 2024

Joint work with Heidi Benham, Andrew DeLapo, Reed Solomon, and Java Darleen Villano, “The Ginsburg–Sands theorem and computability theory”, *Advances in Mathematics* (2024).

The Ginsburg–Sands theorem

Theorem (Ginsburg & Sands, 1979).

Every infinite topological space has an infinite subspace homeomorphic to one of the following topologies on ω:

- indiscrete (only $\emptyset$ and ω are open);
- initial segment (open sets are $\emptyset$, ω, and $[0, n]$ for all $n \in \omega$);
- final segment (open sets $\emptyset$, ω, and $[n, \infty)$ for all $n \in \omega$);
- discrete (all subsets of ω are open);
- cofinite (open sets are $\emptyset$ and all cofinite subsets of ω).

This is a kind of homogeneity property; compare with Ramsey's theorem.

Outline of classical proof

Let X be an infinite topological space. WLOG, assume X is countable.

Define $\sim$ on X by $x \sim y$ if $x \in \mathrm{cl}(y)$ and $y \in \mathrm{cl}(x)$.

- If $[x]_\sim$ is infinite for some x, this is an infinite indiscrete subspace.

So we can assume X is T_0. Define a partial order $\trianglelefteq$ on X by $x \trianglelefteq y$ if $x \in \mathrm{cl}(y)$.

- There is an infinite $\trianglelefteq$-chain/antichain. [CAC]
- Every infinite chain has an infinite ascending/descending sequence. [ADS]
- These are homeomorphic to final/initial segment topologies, respectively.

So can assume X is T_1. If X has no infinite subspace homeomorphic to the cofinal topology, we inductively define an infinite discrete subspace.

Representing topological spaces

Dorais introduced the following framework for dealing with countable, second-countable spaces in the context of computability theory and reverse mathematics.

Definition (Dorais). A CSC space is a triple $(X, \mathcal{U}, k)$ such that:

- $X \subseteq \mathbb{N}$ is infinite.
- $\mathcal{U} = \langle U_i : i \in \mathbb{N} \rangle$ is a sequence of subsets of X containing X and $\emptyset$.
- For all $x \in X$ and $i, j \in \mathbb{N}$, if $x \in U_i \cap U_j$ then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

Thus, $\mathcal{U}$ can be regarded as a countable basis for a topology on X.

Dorais investigated many basic topological facts using this formalization.

Topological properties

In RCA_0, we easily define what it means for a CSC to be indiscrete, discrete, or to have the cofinal topology, as well as many other common topologies properties.

Definition. Let $(X, \mathcal{U}, k)$ be a CSC space.

- X has the initial segment topology if there exists $h : \mathbb{N} \to Y$ such that $U \in \mathcal{U}$ if and only if $U = \emptyset$ or $U = \{h(i) : i \leq j\}$ for some $j \in \mathbb{N}$.
- X has the final segment topology if there exists $h : \mathbb{N} \to Y$ such that $U \in \mathcal{U}$ if and only if $U = X$ or $U = \{h(i) : i \geq j\}$ for some $j \in \mathbb{N}$.

One can give arithmetical definitions for having the initial or final segment topology, and RCA_0 can prove that for the purposes of finding subspaces, these are equivalent [BDDSV].

Statements of principles

Definitions. The following are defined in RCA_0.

GS. Every infinite CSC space has an infinite subspace that is indiscrete, has the initial segment topology, has the final segment topology, is discrete, or has the cofinal topology.

wGS. Every infinite CSC space has an infinite subspace that is indiscrete, has the initial segment topology, has the final segment topology, or is T_1.

GST_1. GS restricted to T_1 CSC spaces.

$\mathsf{GS}^{\mathsf{cl}}$. GS restricted to CSC spaces for which the closure operator exists.

$\mathsf{wGS}^{\mathsf{cl}}$. wGS restricted to CSC spaces for which the closure operator exists.

The full theorem

Theorem (BDDSV). The following are equivalent over RCA_0.

- ACA_0.
- For every CSC space, the closure operator exists.
- GS.
- wGS.

Surprisingly, the proof that wGS $\to$ ACA_0 does not use the closure operator. Rather, $\emptyset'$ is coded directly using subspaces with the initial segment topology.

Theorem (BDDSV). The following are equivalent over RCA_0.

- CAC.
- $\mathsf{wGS}^{\mathsf{cl}}$.

A detour: Ramsey's theorem

Recall RT^2_2, which is Ramsey's theorem 2-colorings of pairs.

- $c : [\mathbb{N}]^2 \to 2$ is stable if $(\forall x)[\lim_y c(x,y) \text{ exists}]$.

  SRT^2_2 is the restriction of RT^2_2 to stable colorings.

- S is cohesive for $\langle X_n : n \in \mathbb{N}\rangle$ if $(\forall n)[|S \cap X_n| < \infty \vee |S \cap \overline{X_n}| < \infty]$.

  COH asserts that every family of sets admits an infinite cohesive set.

Theorem (Cholak, Jockusch, & Slaman). Over RCA_0, $\mathsf{RT}^2_2 \leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$.

The decomposition of principles into some kind of "stable" part, and some kind of "cohesive" part has been remarkably fruitful in the RM investigation of combinatorial principles.

The T_1 case

Theorem (BDDSV). Every computable infinite T_1 CSC space has an infinite Δ^0_2 discrete subspace, or a computable subspace with the cofinal topology.

(We will see that this asymmetry is essential.)

Theorem (BDDSV). Over RCA_0, $\mathsf{GST}_1 \rightarrow \mathsf{ADS}$.

Definition. A T_1 CSC space $(X, \mathcal{U}, k)$ is stable if

$$(\forall x)[\{x\} \in \mathcal{U} \vee (\forall U \in \mathcal{U})[x \in U \rightarrow U =^* X]].$$

- SGST_1 is the restriction of GST_1 to stable spaces.

Theorem (BDDSV). Over RCA_0, $\mathsf{GST}_1 \leftrightarrow \mathsf{SGST}_1 + \mathsf{COH}$.

Subset principles

Definition. Let Γ be a complexity class. Γ-Subset asserts that every Γ-definable subset of $\mathbb{N}$ (which need not exist) has an infinite subset in it or its complement (that does exist).

Theorem (Cholak, Jockusch, & Slaman; Chong, Lempp, & Yang).
Over RCA_0, $\mathsf{SRT}^2_2 \leftrightarrow \Delta^0_2$-Subset. [$\Delta^0_2$-Subset is more commonly called D^2_2]

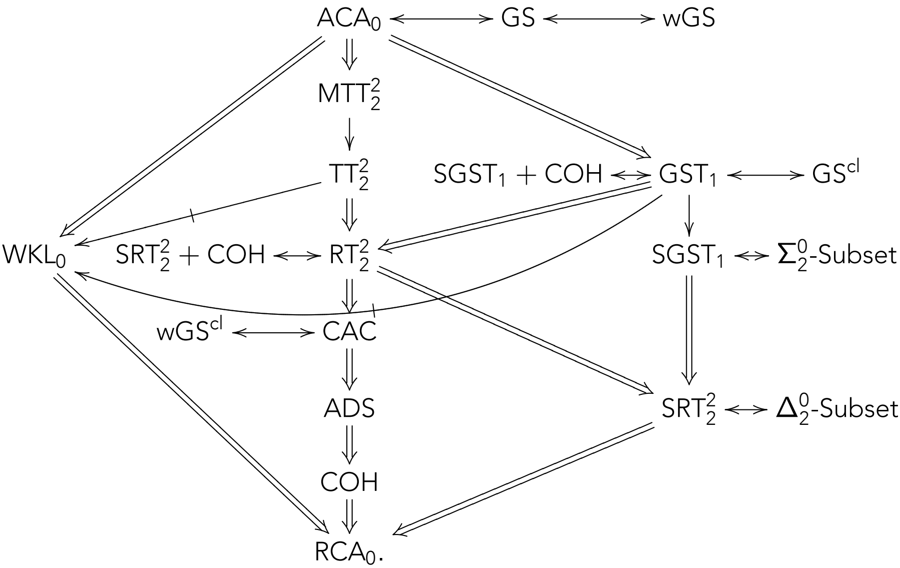
Theorem (BDDSV). Over RCA_0, $\mathsf{SGST}_1 \leftrightarrow \Sigma^0_2$-Subset.

Corollary. $\mathsf{SGST}_1 \to \mathsf{SRT}^2_2$.

Corollary. $\mathsf{GST}_1 \to \mathsf{RT}^2_2$.

Corollary. $\mathsf{GST}_1 \leftrightarrow \mathsf{GS}^{\mathrm{cl}}$.

Between ACA_0 and RT^2_2



Non-implications

Do any of $\mathsf{ACA}_0 \to \mathsf{GST}_1$ and $\mathsf{GST}_1 \to \mathsf{RT}^2_2$ reverse? No.

Theorem (BDDSV). There is an ω-model of GST_1 that does not contain $\emptyset'$.

Corollary. GST_1 does not imply ACA_0 over RCA_0.

This uses so-called strong cone avoidance of RT^1_2 [Dzhafarov & Jockusch] and the equivalence $\mathsf{SGST}_1 \leftrightarrow \Sigma^0_2$-Subset.

Theorem (BDDSV).
There is an ω-model satisfying Σ^0_2-Subset but not Δ^0_2-Subset.

Corollary. RT^2_2 does not imply GST_1 over RCA_0 (even ω-models).

This is a rather intricate forcing argument, using a Σ^1_1 preservation property.

Open questions

Question. Does Σ^0_2-Subset imply COH?

The question of whether Δ^0_2-Subset $\to$ COH was a longstanding question in reverse math [answered by Chong, Slaman, & Yang, and by Monin & Patey].

Question. Is Σ^0_2-Subset Π^1_1-conservative over Δ^0_2-Subset?

Question.

- Does Σ^0_n-Subset $\to$ Δ^0_{n+1}-Subset for any $n \geq 3$?
- Does Δ^0_n-Subset $\to$ Σ^0_n-Subset for any $n \geq 3$?

The answer is "no" for $n = 2$, and it cannot be "yes" to both questions.

Question. How does GST_1 relate to other principles that lie strictly in-between ACA_0 and RT^2_2?

Thank you for your attention!