Semi-periodic Functions and the Scott Analysis of Linear Orderings

David Gonzalez

UC Berkeley

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Outline

- The definitions and history of Scott analysis
- Classification for linear orderings
- A recent construction using Semi-periodic functions

$\mathcal{L}_{\omega_1,\omega}$ formulas and their complexity

- $\mathcal{L}_{\omega_1,\omega}$ is infinitary logic; it extends first order logic by allowing countable conjunctions and disjunctions.
- $\varphi \in \mathcal{L}_{\omega_1,\omega}$ is in $\Sigma_0^{\mathrm{in}} = \Pi_0^{\mathrm{in}}$ if it is quantifier free and has no infinitary disjunctions or conjunctions.
- For $\alpha \in \omega_1$, φ is $\Sigma_\alpha^{\mathrm{in}}$ if $\varphi = \bigvee\!\!\!\bigvee_i \exists \bar{x} \psi_i(\bar{x})$ for $\psi_i \in \Pi_\beta^{\mathrm{in}}$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, φ is Π_α^{in} if $\varphi = \bigwedge\!\!\!\bigwedge_i \forall \bar{x} \psi_i(\bar{x})$ for $\psi_i \in \Sigma_\beta^{\mathrm{in}}$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, φ is d-$\Sigma_\alpha^{\mathrm{in}}$ if $\varphi = \psi \wedge \chi$ for $\psi \in \Sigma_\alpha^{\mathrm{in}}$ and $\chi \in \Pi_\alpha^{\mathrm{in}}$

$\mathcal{L}_{\omega_1,\omega}$ formulas and their complexity

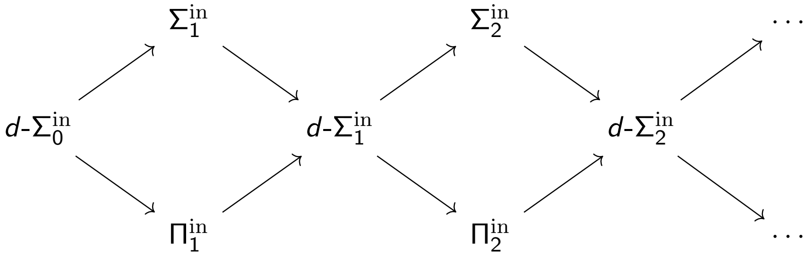
- For two models M, N we say $M \leq_\alpha N$ if $\Pi^{\mathrm{in}}_\alpha - \mathrm{Th}(M) \subseteq \Pi^{\mathrm{in}}_\alpha - \mathrm{Th}(N)$.
- Note that $M \geq_\alpha N$ if and only if $\Sigma^{\mathrm{in}}_\alpha - \mathrm{Th}(M) \subseteq \Sigma^{\mathrm{in}}_\alpha - \mathrm{Th}(N)$.
- We put $M \equiv_\alpha N$ if both of the above hold.

Scott Complexity

Theorem: [Scott] For every countable structure M there is a sentence $\varphi \in \mathcal{L}_{\omega_1,\omega}$ such that $N \cong M \iff N \models \varphi$.

Definition: A φ as in the theorem statement is called a *Scott sentence*.

Definition: *The Scott complexity* (SC) of M is the least among $\{\Sigma^{\mathrm{in}}_\alpha, \Pi^{\mathrm{in}}_\alpha, d\text{-}\Sigma^{\mathrm{in}}_\alpha\}_{\alpha\in\omega_1}$ such that M has a Scott sentence of said complexity.



Robustness and Scott Rank

Definition: [Montalbán] The parametrized *Scott rank* (pSR) of M is the least $\alpha \in \omega_1$ such that M has a $\Sigma^{in}_{\alpha+2}$ Scott sentence.

Definition: [Montalbán] The unparametrized *Scott rank* (uSR) of M is the least $\alpha \in \omega_1$ such that M has a $\Pi^{in}_{\alpha+1}$ Scott sentence.

Theorem: [Montalbán] The (un)parameterized Scott rank of M is the $\alpha \in \omega_1$ such that M the automorphism orbits of all tuples in M are definable in a $\Sigma^{\mathrm{in}}_{\alpha}$ way with(out) parameters. There are also *many* other equivalent statements.

Scott analysis

For $T \in \mathcal{L}_{\omega_1,\omega}$ let

$$I_{SC}(T,\Gamma) = |\{M : M \models T \wedge \mathrm{SC}(M) = \Gamma\}|,$$

$$I_u(T,\alpha) = |\{M : M \models T \wedge \mathrm{uSR}(M) = \alpha\}|,$$

$$I_p(T,\alpha) = |\{M : M \models T \wedge \mathrm{pSR}(M) = \alpha\}|.$$

Scott analysis generally refers to any inquiry into the behavior of the above functions.

Linear orderings

Between G. and Rosseger 2023 and G., Harrison-Trainor and Ho 2024 we fully characterized the I_{SC} function for linear orderings.

1. $I_{SC}(LO, \Pi_n) = 1$ if $n \leq 2$
2. $I_{SC}(LO, \Sigma_n) = 0$ if $n \leq 3$
3. $I_{SC}(LO, \text{d-}\Sigma_n) = \aleph_0$ if $n \leq 4$
4. $I_{SC}(LO, \Sigma_4) = \aleph_0$
5. $I_{SC}(LO, \Pi_3) = \aleph_0$
6. $I_{SC}(LO, \Gamma) = 2^{\aleph_0}$ otherwise

The Relationship of the Concepts

Alvir, Greenberg, Harrison-Trainor and Turetsky (AGHTT) showed that Scott sentence complexity is related to Montalbán's Scott ranks.

SC	pSR	uSR	complexity of parameters
$\Sigma^{\mathrm{in}}_{\alpha+2}$	α	$\alpha+2$	$\Pi^{\mathrm{in}}_{\alpha+1}$
$d\text{-}\Sigma^{\mathrm{in}}_{\alpha+1}$	α	$\alpha+1$	$\Pi^{\mathrm{in}}_{\alpha}$
$\Pi^{\mathrm{in}}_{\alpha+1}$	α	α	none
α limit			
$\Sigma^{\mathrm{in}}_{\alpha+1}$	α	$\alpha+1$	$\Pi^{\mathrm{in}}_{\alpha}$
$\Pi^{\mathrm{in}}_{\alpha}$	α	α	none

For limit α, $\Sigma^{\mathrm{in}}_{\alpha}$ and $d\text{-}\Sigma^{\mathrm{in}}_{\alpha}$ are not possible.
Notice the limit case is left ambiguous in their work.

Limit levels: History of the Mystery

- In 1983, A. Miller gave examples of all possible Scott complexities except for $\Sigma_{\lambda+1}$.
- 38 years later AGHTT gave an example of Scott complexity $\Sigma_{\lambda+1}$.
- The example is very pretty, but quite complex.
- $\Sigma_{\lambda+1}$ was left open by G. and Rossegger - G., Harrison-Trainor and Ho filled this gap with a new construction with surprisingly life

The Construction: Semi-periodic Functions

Definition: A function $f : \mathbb{Z} \to \mathbb{N}$ is *semi-periodic* if for all n $f_n := \min(f, n)$ is periodic but f itself is not periodic.

Example: Let $\sigma_0 = 0$ and $\sigma_{i+1} = \sigma_i^\frown(i+1)^\frown\sigma_i$. Limits of these finite strings produce semi-periodic function.

$\cdots$,4,0, 1, 0, 2, 0, 1, 0, 3,0, 1,0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, $\cdots$

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The Construction: λ-mixable pairs

Definition

An ordered pair $(\{L_i\}_{i\in\omega}, K)$ of a sequence of linear orderings $\{L_i\}_{i\in\omega}$ and a single linear ordering K is called a *λ-mixable pair* if the following properties hold for some non-zero fundamental sequence $\delta_i \to \lambda$:

1. $\mathrm{uSR}(K) < \lambda$
2. $\mathrm{uSR}(L_i) < \lambda$
3. $L_i \equiv_{\delta_i} L_{i+1}$
4. $L_i \not\equiv_{\delta_{i+1}} L_{i+1}$
5. any finite alternating sum $1 + L_{a_0} + 1 + K + 1 + L_{a_1} + 1 + K + \cdots + 1 + L_{a_n}$ has intervals isomorphic to K only within the written K blocks (or as the entire written K block).

The Construction: λ-mixable pair examples

- $L_i = \omega^{\delta_i}$ and $K = \zeta$
- $L_i = \zeta^{\delta_i}$ and $K = \omega$
- $L_i = \sum_{n<i}(n + \zeta^{\delta_n}) + \sum_{n \geq i}(n + \zeta^{\delta_i})$ and $K = \omega$.

The Construction

Given any λ-mixable pair $(\{L_i\}_{i\in\omega}, K)$ and semi-periodic function $f : \mathbb{Z} \to \mathbb{N}$,

$$L_f = \sum_{n\in\mathbb{Z}} (1 + L_{f(n)} + 1 + K)$$

has Scott complextiy $\Sigma_{\lambda+1}$.

Example:

$$\cdots + \omega + \zeta + \omega^2 + \zeta + \omega + \zeta + \omega^3 + \zeta + \omega + \zeta + \omega^2 + \zeta + \omega + \cdots$$

Going Further: The Structure of Semi-periodic Functions

Definition: $fE_{\mathbb{Z}}g$ if there is an $n \in \mathbb{Z}$ such that for all $m \in \mathbb{Z}$ $f(m) = g(m+n)$.

Definition: $fE_{fin}g$ if for all n $f_n E_{\mathbb{Z}} g_n$.

Translation: $fE_{\mathbb{Z}}g$ if and only if $L_f \cong L_g$. $fE_{fin}g$ if and only if $L_f \equiv_\lambda L_g$.

$\Sigma_{\lambda+1}$ gives back

Surprising: We now know more about $\Sigma_{\lambda+1}$ than $\Sigma_{\alpha+1}$ for non-limit α.

Proposition: There are continuum many Scott complexity $\Sigma_{\alpha+1}$ in some $\equiv_\alpha$ class.

Proposition: There is a $\equiv_\alpha$ class with only structures of Scott complexity $\Sigma_{\alpha+1}$.

Proposition: There is a rigid structure of Scott complexity $\Sigma_{\alpha+1}$.

Thank you!

Similar and Simple

We found continuum many $\equiv_\lambda$-equivalent linear orderings all with Scott complexity $\Sigma_{\lambda+1}$.

Combinatorial solution: There are continuum many E_{fin} classes in a single $E_{\mathbb{Z}}$ class. In fact, $E_0 \leq_B E_{\mathbb{Z}}|_{[b]_{E_{fin}}}$.

Similar and Simple

Proof idea: An increasing enumeration of a non-zero set $A \in 2^\omega$ gives rise to a system of embeddings $\sigma_i \to \sigma_j$.

$$\sigma_0 \hookrightarrow \sigma_{A(1)} \hookrightarrow \sigma_{A(2)} \hookrightarrow \sigma_{A(3)} \hookrightarrow \cdots$$