A Return to Degree Spectra of Relations on a Cone

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The setting throughout this talk is:

- $\mathcal{A}$ is a (computable) structure
- R is an additional (computable) relation on $\mathcal{A}$.

Definition

The degree spectrum of R is

$$\mathrm{dgSp}(R) = \{d_T(R^{\mathcal{B}}) : \mathcal{B} \text{ is a computable copy of } \mathcal{A}\}.$$

Suppose that $\mathcal{A}$ and R are "natural". What are the possible degree spectra?

In my thesis, I considered degree spectra on a cone, with on a cone capturing the intuitive idea that $\mathcal{A}$ and R are natural.

One particularly interesting example is the case where the structure is $\mathcal{A} = (\mathbb{N}, <)$. This example had been considered by Wright and myself, and more recently Bazhenov, Kalociński, and Wrocławski.

This talk is about recent work with Jad Damaj in which we solved a question I had not been able to solve in my thesis: we produce examples of natural relations on $\mathbb{N}$ which have surprising degree spectra on a cone.

Degree spectra relative to a cone

Let $\mathcal{A}$ be a computable structure and R a relation on $\mathcal{A}$.

Definition

The *degree spectrum of R below the degree* $\mathbf{d}$ is

$$\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{d(R^{\mathcal{B}}) \oplus \mathbf{d} : \mathcal{B} \cong \mathcal{A} \text{ and } \mathcal{B} \leq_T \mathbf{d}\}$$

Recall that a cone is a set of Turing degrees

$$C_{\mathbf{d}} = \{\mathbf{c} : \mathbf{c} \geq_T \mathbf{d}\}.$$

Given enough determinacy, any set of Turing degrees either contains a cone or is disjoint from a cone.

We think of sets containing a cone as being "large", and the complement as being "small".

Relativised degree spectra

Let $\mathcal{A}$ and $\mathcal{B}$ be structures and R and S relations on $\mathcal{A}$ and $\mathcal{B}$ respectively.

For any degree $\mathbf{d}$, either $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}}$ is equal to $\mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$, one is strictly contained in the other, or they are incomparable. By Borel determinacy, exactly one of these happens on a cone.

Definition (Montalbán)

The degree spectrum of $(\mathcal{A}, R)$ on a cone is equal to that of $(\mathcal{B}, S)$ if we have equality on a cone, and similarly for containment and incomparability.

Example

If $\mathcal{A} = (\mathbb{N}, <)$ and R is the empty unary relation $R =$, then for all degrees $\mathbf{d}$ we have $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{\mathbf{d}\}$.

Example

If $\mathcal{A} = (\mathbb{N}, <)$ and R is the successor relation, then for all degree $\mathbf{d}$ we have $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} =$ degrees c.e. in and above $\mathbf{d}$.

These are the two minimal examples.

Theorem (Harizanov)

One of the following is true for all degrees $\mathbf{d}$ on a cone:

1. $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{\mathbf{d}\}$, *or*
2. $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \supseteq$ *degrees c.e. in and above* $\mathbf{d}$.

D.c.e. relations

Theorem (HT)

There is a computable structure $\mathcal{A}$ and relatively intrinsically d.c.e. relations R and S on $\mathcal{A}$ with the following property:

$$\text{for any degree } \mathbf{d},\ \mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \text{ and } \mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}} \text{ are incomparable.}$$

Corollary (HT)

There are two degree spectra on a cone which are incomparable, each contained within the d.c.e. degrees and containing the c.e. degrees.

Theorem (HT)

Let $\mathcal{A}$ be a structure and R a relation on $\mathcal{A}$. Then one of the following is true relative to all degrees on a cone:

1. $\mathrm{dgSp}(\mathcal{A}, R) \subseteq \Delta^0_2$, *or*
2. $2\text{-}CEA \subseteq \mathrm{dgSp}(\mathcal{A}, R)$.

Theorem (Wright)

For every relation R on $(\mathbb{N}, <)$, the degree spectrum is either:

- *the computable degrees,*
- *the c.e. degrees,*
- *the Δ^0_2 degrees, or*
- **intermediate** *between the c.e. degrees and the Δ^0_2 degrees.*

The successor relation often plays an important role.

Theorem (HT)

If a relation R on $(\mathbb{N}, <)$ *is intrinsically* α*-c.e. on a cone, then its degree spectrum on a cone is either the computable degrees or the c.e. degrees.*

Question

Is there a relation R whose degree spectrum is intermediate (on a cone)?

Theorem (Bazhenov, Kalociński, and Wrocławski)

There is a total computable function whose degree spectrum (not on a cone) strictly contains all c.e. degrees and is strictly contained in the Δ^0_2 degrees.

This total computable function is not a natural one.

The proof is by a priority argument:

For each $e, e_1, e_2, n \in \omega$, we have the following requirements:

$$\mathcal{I}_e : I \not\leq \Phi_e^J, \quad \mathcal{J}_e : J \not\leq \Phi_e^I, \quad \text{and} \quad R_{\langle e_1, e_2, n\rangle} : \Phi_{e_1}^{\Gamma_{f_\mathcal{A}}} \not\simeq W_n \vee \Phi_{e_2}^{W_n} \not\simeq \Gamma_{f_\mathcal{A}},$$

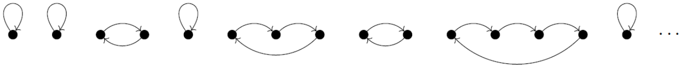
where $\Gamma_{f_\mathcal{A}}$ is the graph of $f_\mathcal{A}$. The non-c.e. degree requirements are based on [5, p. 195].

On a cone, the degree spectrum is the c.e. degrees.

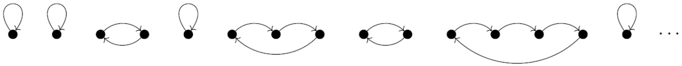
Theorem (Damaj, HT)

There are relations on $(\omega, <)$ whose degree spectrum on a cone is strictly between the c.e. degrees and the Δ_2^0 degrees.

One example is:



This follows a simple pattern; the hard work comes after the construction.



Let l_n be the loop of length n. The function is described by:

$$l_1 l_1 l_2 l_1 l_3 l_2 l_4 l_1 l_5 l_2 l_6 l_3 l_7 l_1 l_8 \ldots.$$

- The blocks in odd positions follow the pattern $l_1 l_2 l_3 l_4 \ldots$ enumerating the natural numbers in increasing order
- The blocks in even positions $l_1 l_1 l_2 l_1 l_2 l_3 \ldots$ are an enumeration of all of the natural numbers such that each number occurs infinitely many times.
- Every block appears infinitely many times, but any pair of blocks appears adjacent to each other only once.

We do not have a good characterization of these intermediate degree spectra. Using our method of constructing these relations R, for any computable listing of indices for Δ^0_2 sets, the degree spectrum of R contains some degree which is not in the listing.

Theorem (Damaj, HT)

For every computable ordinal α, there is a relation R on $(\mathbb{N}, <)$ whose degree spectrum on a cone is intermediate but contains all α-c.e. degrees.

Thanks!