Word problems of groups as ceers

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A brief history

1880 Dyck: study group presentations $\approx$ Combinatorial Group Theory

1911 Dehn: word, conjugacy, and isomorphism problems for groups

1930s Godel, Church, Turing, Kleene, Post: Computability Theory

1947 Post: f.p. semigroup with unsolvable word problem

1950 Turing: f.p. cancellative semigroup with unsolvable word problem

1950s Novikov; Boone: f.p. group with unsolvable word problem

1950s Adian; Rabin: isomorphism problem for f.p. groups is unsolvable

Dehn's problems

1911 Dehn: word, conjugacy, and isomorphism problems for groups

Word problem

Fix a presentation of a group, determine if two given words are equal.

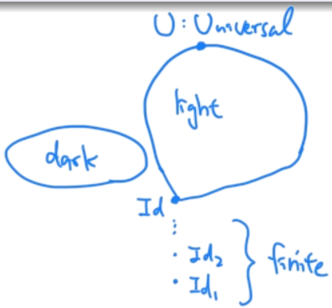
- A group presentation $\langle S \mid R \rangle$ is
  - *finitely presented (f.p.)* if S and R are both finite.
  - *finitely generated (f.g.)* if S is finite (and R is c.e.).
  - *computably enumerable (c.e.)* if S is computable and R is c.e.
- Every group in this talk comes with a c.e. presentation.

Computably enumerable equivalence relations

Classically, decision problems are considered as subsets of ω and compared using *Turing reduction*.
Word, conjugacy, and isomorphism problems are naturally *computably enumerable equivalence relations (ceers)*.

Definition

A ceer E is *reducible* to another ceer F, denoted $E \leq F$, if there is a computable function f such that $iEj \Leftrightarrow f(i)\ F\ f(j)$ for every i, j.



Word and conjugacy problems

Theorem (Myasnikov and Osin '11)

There is a f.g. group G that is algorithmically finite, i.e., for every infinite c.e. set S there are $x, y \in S$ such that $x =_G y$. These are exactly the dark word problems.

Conjugacy problem

Fix a presentation of a group, determine if two given words are conjugate, i.e., given x, y, determine if $\exists z, z^{-1}xz = y$.

Corollary

There is a f.g. group G such that $WP(G) \nleq CP(G)$.

Word problems

Theorem (Fridman; Clapham; Boone; Bokut '60s)

Every c.e. degree is realized by word problems of f.p. groups.

Question

Which ceers are realized by word problems of c.e./f.g./f.p. groups?

Proposition

There are ceer degrees that are not realized by word problems of groups.

Theorem (Della Rose, San Mauro, and Sorbi '23)

There are groups whose word problem degree cannot be realized by f.g. groups.

Failure of classical proofs

The proofs that every c.e. degree is realized by word problems of f.p. group use HNN and Higman embedding theorems, which in turn use free products and HNN extensions.

Theorem (AH)

There is a non-universal $WP(G)$ such that the word problem of the free product $WP(G * H)$ is universal for every nontrivial H.

Question

Can G be f.g. or f.p.?

Ceers realized by word problems

Theorem (AHTH)

Any nontrivial Σ^0_3 index set of ceers is Σ^0_3-complete.

Corollary (AHTH)

The set of ceers realized by the word problems of c.e./f.g./f.p. groups are Σ_3-complete.

Degrees realized by word problems

Question

What is the structure of degrees realized by word problems?

Theorem (AHSM)

There is a non-universal E such that every $WP(G) \geq E$ is universal.

Theorem (AHSM)

There are $E < WP(G)$ such that for any $E \leq WP(H) \leq WP(G)$, we have $H \cong G$.

Theorem (AHSM)

For every non-universal $WP(G)$, there is some $WP(G) < WP(H) < U$.

Isomorphism problem

Isomorphism problem

Determine if two given presentations define isomorphic groups.

Theorem (Miller '71)

The isomorphism problem for f.p. groups is a universal ceer.

Theorem (AHTH)

The isomorphism problem for c.e./f.g. groups is a Σ_3-complete equivalence relation.