Separating notions in effective topology and analysis

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Computable analysis

Definition (Mal'cev, Rabin, 60's)

A countable structure is computable if its domain and all operations and relations are uniformly computable.

We want to study computable uncountable structures. To apply tools of classical computability, we only consider structures that are *countably based*.

Computable analysis has laid the framework and provided intuition on working with "uncountable" effective objects.

Analogously, a Polish space can be said to be computable if it is the completion of a *computable metric space*.

Computable metric spaces

Definition (Ceitin 1959, Moschovakis 1964)

A computable metric space (S, d) consists of a countable set $S = \{c_0, c_1, \cdots\}$ and $d : \mathbb{N}^2 \mapsto \mathbb{R}$ such that $d(c_i, c_j)$ is a computable real number, and $\overline{(S, d)}$ is Polish with metric induced by d.

A computable metric structure (S, d) can also be viewed as a computable presentation of the underlying topological space.

Via this representation, we can talk about computable isometries, associate degrees to points, etc.

We can extend computability to a countably based (not necessarily metrizable) topological space.

Computable topological spaces

Definition

A topological space is effectively second countable (or simply *computable*) if there is a countable base $\{B_i\}_{i\in\omega}$, and a computable function f such that $B_i \cap B_j = \bigcup_{k\in W_{f(i,j)}} B_k$, and where "$B_i \cap B_j \neq \emptyset$" is c.e.

These definitions allow one to study many effective aspects of non-countable countably based spaces.

The countable presentations of these spaces are "point-free", and is a very weak notion.

Presentability of a Polish space

Degree spectra

(Selivanov) Turing degree spectra of topological spaces.

(Hoyrup, Kihara, Selivanov) Gave examples of topological degree spectra.

(Clanin, McNicholl, Stull) Studied the isometric types of a Polish space.

We want to consider the homeomorphism type of a Polish space - A Polish space is not always homeomorphic to a computable Polish space. The difficulty of classifying homeomorphism types (of metric spaces) is that the metric is not preserved.

Polish spaces with no computable copy

A basic question: Is there a Polish space that is not homeomorphic to a computable Polish space?

Theorem (Greenberg, Montalban)

Every countable hyperarithmetical compact Polish space has a computable copy.

Theorem (Hoyrup-Kihara-Selivanov, Harrison-Trainor-Melnikov-N, Lupini-Melnikov-Nies)

There is a Δ^0_2 presentable Polish space which is not homeomorphic to a computable Polish space.

Note: The same question is trivial up to isometry. E.g. $([0, \alpha], |\cdot|)$ where α is a right-c.e. non-computable real.

Star spaces

We introduce the (very useful) notion of a *star space*:

Definition

An n-star ($n \geq 3$) is a (homeomorphic) copy of n copies of $[0, 1]$ with all the left endpoints identified.
A *star space* X is the disjoint union of countably many stars.

This gives a topological invariant that can be recognised relatively easily:

Lemma (Harrison-Trainor-Melnikov-N)

Given a computable Polish presentation X of a star space, it is Σ^0_3 to tell if X contains a star of size $\geq n$, uniformly in X and n.

Star spaces

Theorem (Hoyrup-Kihara-Selivanov, Harrison-Trainor-Melnikov-N, Lupini-Melnikov-Nies)

There is a Δ^0_2 presentable Polish space which is not homeomorphic to a computable Polish space.

Proof.

In a computable Polish presentation X of a star space, the set $\{n \mid X \text{ contains an } n\text{-star}\}$ is limitwise-monotonic relative to $\emptyset''$.

Take Σ^0_3 sets S, T such that $S \setminus T$ is not limitwise-monotonic relative to $\emptyset''$.

Build a Δ^0_2 Polish star space that realises $S \setminus T$. □

Lower- and upper-semicomputable Polish spaces

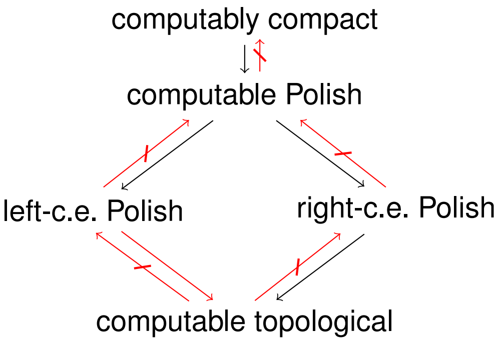
We refine Δ^0_2-presentability of a Polish space:

Definition

A *right-c.e.* metric space (S, d) consists of a countable set $S = \{c_0, c_1, \cdots\}$ and $d : \mathbb{N}^2 \mapsto \mathbb{R}$ such that $d(c_i, c_j)$ is a right-c.e. real number, uniformly in i, j. Same for a *left-c.e.* metric space.

These are the analogues of c.e. and co-c.e. presentations in effective (countable) algebra, e.g. discrete topological groups, Boolean algebras.

Variations on effective presentability of a Polish space



Arrows represent trivial implications, up to homeomorphism.

Each right c.e. Polish space is computable topological with the obvious basis induced by the metric $\{B(c_i, 2^{-n}) \mid c_i \in S, n \in \omega\}$.

computable Polish $\not\Rightarrow$ computably compact

Theorem (Hoyrup-Kihara-Selivanov, Lupini-Melnikov-Nies, Koh-Melnikov-N)

There is a compact computable Polish space with no computably compact presentation.

Proof.

In a computably compact Polish presentation X of a star space, complexity is reduced by a jump. □

Right c.e. Polish $\not\Rightarrow$ computable Polish

Theorem (Harrison-Trainor, Melnikov, N)

A countable Boolean algebra has a computable presentation $\Leftrightarrow$ its dual Stone space has a computable Polish presentation.

Theorem (Bazhenov, Harrison-Trainor, Melnikov)

A countable Boolean algebra has a c.e. copy iff its dual Stone space has a right c.e. computably compact Polish presentation.

Proposition (Melnikov, N)

Every left c.e. Stone space is homeomorphic to a computable Polish space.

Combining with Feiner's result, one can obtain a right c.e. Stone space with no computable (left c.e.) Polish presentation.

Computable topological $\not\Rightarrow$ Arithmetical Polish

Theorem (Melnikov, N)

For any set X there is a computable topological (locally compact) Polish space with no X-computable Polish presentation.

Computable topological presentations are point-free: S^n and $\mathbb{R}^n$ share a same computable topological presentation.

Choose S^n or $\mathbb{R}^n$ depending on whether or not $n \in X$.

A remarkable fact is that the above fails for abelian or locally compact *topological groups*.

Computable topological groups

A computable topological group is a computable topological Hausdorff space where the group operations are computable.

Theorem (Koh, Melnikov, N)

Each computable topological group G admits a right c.e. compatible left-invariant metric. If G is effectively locally compact, then the metric is effectively proper.

Corollary

If G is either abelian or locally compact, then computable topological $\Leftrightarrow$ right c.e. Polish presentable.

Left c.e. Polish $\not\Rightarrow$ Computable topological

Recall that every right c.e. Polish space is (via the same basis) computable topological.

It is also obvious that the basis corresponding to a left c.e. Polish space does not necessarily give a computable topological presentation.

Theorem (Bazhenov, Melnikov, N)

Every Δ^0_2 Polish space has a basis given by a computable topological presentation.

Further separations

Theorem (Koh, Melnikov, N)

There is a Δ_2^0 compact Polish space that has no left-c.e Polish nor a right-c.e. Polish copy.

Approximable from below $+$ approximable from above $=$ computable?

Theorem (Koh, Melnikov, N)

There is a locally compact Polish space that has a left-c.e. Polish presentation and a right-c.e. Polish presentation, but no computable Polish presentation.

Questions

Question

Does every (effectively) compact left-c.e. Polish space have a computable (right c.e.) Polish copy?

Question

Does every Δ^0_3 Polish space admit a computable topological presentation? What about arithmetical Polish spaces?

Question

Is there a Polish space with no computable topological presentation?

Thank you