Complexity Notions on Monoids and Pointwise Dimension

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ASL Special Session on Computability Theory

A Tale of Two Fractals

Cantor sets

- "Deterministic" fractals.
- Determined by e.g., splitting ratio and gaps between intervals.

Point fractals

- "Random" fractals.
- Membership is determined via a property of its points.

Point fractals via effective dimension

Given $\alpha \in [0, 1]$, let

$$K_\alpha = \{X \in 2^\omega \colon \liminf_{n\to\infty} \frac{K(X\lceil_n)}{n} = \alpha\}$$

Cai & Hartmanis (1994):

$$\dim_H K_\alpha = \alpha$$

Early version of **point-to-set principle**.

Point fractals via effective dimension

$$\dim_H K_\alpha = \alpha$$

The Cai-Hartmanis theorem (lower bound) can be proved by

1. finding a suitable Cantor set sitting inside K_α and using the **mass distribution principle**, or
2. using the point-to-set principle.

Point sets in Euclidean space

Identifying elements of 2^ω with binary expansions of reals, Cai and Hartmanis actually proved the theorem for Hausdorff dimension in $\mathbb{R}$.

Hypothesis: The sets K_α lie at the extreme end of a whole family of similarly behaved point fractals in Euclidean space.

Point fractals: Irrationality

The **irrationality exponent** of a real number x is defined as

$$\delta(x) = \sup \left\{ \delta \colon \exists^\infty p, q \; \left| x - \frac{p}{q} \right| < \frac{1}{q^\delta} \right\}.$$

Every irrational number has irrationality exponent ≥ 2 [Dirichlet].

A Liouville number is defined by the property $\delta(x) = \infty$.

Other examples:
– $\delta(e) = 2$
– $\delta(\pi) \leq 7.60630853$

Point fractals: Jarnik-Besicovitch

Jarník (1931), and independently Besicovitch (1934), showed

$$\dim_{\mathrm{H}}\{x\colon \delta(x) \geq \delta\} = \frac{2}{\delta}$$

Jarník's proof actually shows that

$$\dim_H\{x\colon \delta(x) = \delta\} = \frac{2}{\delta}.$$

Effective Dimension

The irrationality exponent reflects how well a real can be approximated by rational numbers.

Information theoretically: Think of (p, q) as a description of a real with respect to a very simple decoder: $(p, q) \mapsto p/q$.

The **effective dimension** [Lutz] reflects how well a real can be approximated by arbitrary effective decoders:

$$\dim_{\mathrm{H}}(x) = \liminf_{n \to \infty} \frac{K(x \upharpoonright n)}{n}$$

(The Kolmogorov complexity characterization is due to Mayordomo.)

Effective Dimension and Irrationality Exponent

For a random real $x \in [0, 1]$, p/q cannot give significantly more than $2 \log q$ bits of information about x.

Hence almost every real has irrationality exponent 2.

If $x \in (0, 1)$ is Liouville, on the other hand, for every n there exist p/q such that $2 \log q$ bits of information give us $n \log q$ bits of x

Hence the effective dimension of a Liouville number is 0 [Staiger, 2002]

This line of reasoning can be generalized to obtain

$$\dim_{\mathrm{H}}(x) \leq \frac{2}{\delta(x)} \qquad \text{[Calude \& Staiger, 2013].}$$

Diophantine complexity

We can reformulate the irrationality exponent as an "effective dimension".

For $x \in \mathbb{R}$, let

$$K_n(x) = \min\{K(p/q) \colon |x - p/q| \leq 2^{-n}\}.$$

(the Kolmogorov complexity at precision n).

Then

$$\dim_{\mathrm{H}}(x) = \liminf_n \frac{K_n(x)}{n}$$

[Lutz & Mayordomo, 2008]

Diophantine complexity

Similarly, let

$$D_n(x) = \min\{2 \log q \colon \exists p\ |x - p/q| \leq 2^{-n}\}.$$

(the Diophantine complexity at level n).

Then

$$\delta(x) = \liminf_n \frac{D_n(x)}{n}$$

Effective Dimension and Irrationality Exponent

Question: Are these two isolated examples, or are they instances of a more general phenomenon?

Hausdorff Dimension vs Irrationality Exponent

Theorem [Becher, R., & Slaman, 2018]

Let $\delta \geq 2$. For every $\beta \in [0, 2/\delta]$ there is a Cantor-like set E such that $\dim_H(E) = \beta$ and for the uniform measure on E, almost all real numbers have irrationality exponent δ.

Theorem [B-R-S, 2018]

Let $\delta \geq 2$. For every $\beta \in [0, 2/\delta]$ there is a Cantor-like set E such that for the uniform measure on E, almost all real numbers have irrationality exponent δ and effective dimension β.

How much compression?

Question: How much ”compression” is needed to obtain point fractals that exhibits the Jarnik-Besicovitch-Cai-Hartmanis stratification with respect to Hausdorff dimension?

Generalized Complexity Measures

Let $(M, \cdot, 1)$ be a monoid.

We call a function $C : M \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ a **complexity measure** if the following hold:

(a) $C(1) = 0$

(b) $C(xy) \leq C(x) + C(y)$ for all $x, y \in M$

(c) $C(x) \leq C(xy) + C(y)$ for all $x, y \in M$

(d) $C(xy) = C(yx)$ for all $x, y \in M$

We call the pair (M, C) **C-monoid**.

Examples

– Prefix-free Kolmogorov complexity (up to an additive constant) on set of strings with concatenation (if we restrict (d) to strings of the same length).

– The height map $(p/q) \mapsto \log |q|$.

– Finite measurable partitions of $[0, 1]$ under refinement with entropy.

– Many other examples: the least degree of a polynomial vanishing identically on a variety; invariant metrics on groups (distance from 0), . . .

Suppose M is dense in a metric space (X, d) and for each $r > 0$, the set $M_r = \{m \in M\colon C(m) \leq r\}$ is finite. For $x \in X$, let

$$C_n(x) = \inf\{C(m)\colon d(x, m) \leq 2^{-n}\}$$

and

$$\overline{C}(x) = \sup\{\beta\colon \exists^\infty m\, d(x, m) \leq 2^{-\beta C(m)}\}.$$

Theorem [Cotner & R.]
For all $x \in X \setminus M$,

$$\liminf_n \frac{C_n(x)}{n} = \overline{C}(x)^{-1}.$$

Normal compressors

Normal compressors [Cilibrasi & Vitanyi, 2005]

– Defined on the monoid $2^{<\omega}$.

– Upper-semicomputable, induced by prefix-codes.

– Relaxation: Require inequalities to hold only up to $\log(n)$ term, where n is the maximal length of a string involved in the inequality concerned.

– Compression: Require additionally

- $C(xx) \leq C(x)$
- $C(x) \leq C(xy)$
- $C(xy) + C(z) \leq C(xz) + C(yz)$

Dimension of generalized complexity level-sets

Theorem

Let C be a normal compressor on $2^{<\omega}$. Then, for any $\alpha \in [0,1]$,

$$\dim_H \left\{ x \colon \liminf_{n\to\infty} \frac{C(x\lceil_n)}{n} = \alpha \right\} = \alpha.$$

Dimension of generalized complexity level-sets

Theorem

Let C be a complexity measure on $2^{<\omega}$ such that for each $r > 0$, the set $M_r = \{m \in M\colon C(m) \le r\}$ is finite. Assume there exists a constant $c > 0$ and a mapping $\pi : 2^{<\omega} \to \mathbb{Q} \cap [0,1]$ such that for any interval $I \subset [0,1]$ there exists a $k_0 > 0$ such that for all $k \ge k_0$ there exist $\exists \sigma_1, \dots \sigma_t \in 2^{<\omega}$ with $\pi(\sigma_i) \in I$ for all $i \le t$ and

(i) $C(\sigma_i) \le \log(k)$ for all $i \le t$,
(ii) $|\pi(\sigma_i) - \pi(\sigma_j)| \ge \frac{1}{k}$ for $i \ne j$, and
(iii) $t \ge c|I|k$

Then, for any $\alpha \in [0,1]$,

$$\dim_H \left\{ x\colon \liminf_{n\to\infty} \frac{C_n(x)}{n} = \alpha \right\} = \alpha.$$

Dimension of generalized complexity level-sets

– Being a normal compressor ensures that the effective C-dimension (in 2^ω) agrees with the generalized irrationality exponent (in $\mathbb{R}$).

– Being a complexity measure together with (i)-(iii) ensures that the image of C under π induces a **regular system** [Baker & Schmidt, 1970].

Fourier dimension

A (Borel) measure μ on $\mathbb{R}$ is an **α-Fourier measure** if there exists a constant c such that for all $x \in \mathbb{R}$,

$$|\widehat{\mu}(x)| \leq c|x|^{-\alpha/2},$$

where $\widehat{\mu}$ is the Fourier-Stieltjes transform of μ.

The **Fourier dimension** of $A \subseteq \mathbb{R}$ is defined as

$$\dim_F A = \sup\{\alpha\colon \exists\alpha\text{-Fourier } \mu, \mu(A) = 1\}.$$

Salem sets

It always holds that $\dim_F \leq \dim_H$, but they can be drastically different.

For example, $\dim_F(\text{middle-third Cantor set}) = 0$.

Sets for which $\dim_F = \dim_H$ are called **Salem sets**.

"Deterministic" constructions of Salem sets were long elusive, until Kaufman showed that the Jarnik-Besicovitch fractal is a Salem set.

Salem sets

"The property of being a Salem measure [...] is deeper than average decay [...] and indicative of the level of the arithmetic structure of the measure in question. Roughly speaking, "random" fractal measures often [are Salem], whereas "structured" ones [such as the middle third Cantor measure, are not]." (Laba, ICM 2014)

Salem sets via complexity measures?

Conjecture: If C is a normal compressor, then

$$\left\{x\colon \liminf_{n\to\infty} \frac{C(x\lceil_n)}{n} = \alpha\right\}$$

is a Salem set.