Existentially closed groups

A bridge between computability theory, model theory, and algebra

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Key Definition

Definition

An existentially closed group *M is a group such that if there is a group $N \geq M$ and quantifier-free $\varphi(\overline{x}, \overline{m})$ with parameter from M such that*

$$N \models \exists \overline{x} \varphi(\overline{x}, \overline{m})$$

then

$$M \models \exists \overline{x} \varphi(\overline{x}, \overline{m})$$

Some history: A "bridge"

Theorem ([Neu73], [Mac72a], [Rip82])

For a finitely generated group G, the following are equivalent:

1. *G has solvable word problem*
2. *G embeds in every existentially closed group*
3. *G is $\exists_1$-isolated; i.e., there is a consistent $\exists_1$-formula $\psi(\overline{x})$ of group theory such that for every quantifier-free $\varphi(\overline{x})$, $T \vdash \forall \overline{x}(\psi(\overline{x}) \to \varphi(\overline{x}))$ iff $G \models \varphi(\overline{g})$*

Theorem ([Mac72a])

Let G and H be finitely generated groups with $H \not\leq_T G$. Then there is an e.c. group M which contains G and omits H.

History, continued

Theorem ([Mac72a])

Let G and H be finitely generated groups with $H \not\leq_T G$. Then there is an e.c. group M which contains G and omits H.

Theorem ([Zie80])

For finitely generated groups G and H, the following are equivalent:

1. *H embeds in every e.c. group that G does*
2. *$H \leq^* G$*

First question

Question

How hard is it to build existentially closed groups?

Here, by “existentially closed group”, I will always mean its quantifier-free diagram.

First answer: the complexity of existentially closed groups

Theorem (S.)

The existence of an existentially closed group is equivalent to $0'$.

Building an existentially closed group 1: Henkin construction

Let C be a countable set of constants and let $\langle \varphi_i(\overline{x}, \overline{c}) : i < \omega \rangle$ be an enumeration of the quantifier-free formulas in $L(C)$. Build the diagram of an existentially closed group as follows:

- Let T_0 consist of the axioms of group theory.
- On odd stages $s = 2t + 1$, add $\varphi_t(\overline{d}, \overline{c})$ to T_s if consistent; otherwise do nothing.
- On even stages $s = 2t$, add tth sentence of $L(C)$ to T_s if consistent; otherwise add negation.

Let $T = \bigcup_s T_s$.

Building existentially closed groups 2: Fraïssé's Theorem

Theorem (Fraïssé's Theorem)

Let $\mathcal{A}$ be a countable collection of isomorphism types of finitely generated structures of some theory T groups satisfying:

1. *$\mathcal{A}$ has the Hereditary Property (HP): for every $B \leq A \in \mathcal{A}$, $B \in \mathcal{A}$.*
2. *$\mathcal{A}$ has the Joint Embedding Property (JEP): for every $A, B \in \mathcal{A}$, there is some $C \in \mathcal{A}$ with $A, B \leq C$.*
3. *$\mathcal{A}$ has the Amalgamation Property (AP): for every $A, B, C \in \mathcal{A}$, with $C \leq A, B$, there is a $D \in \mathcal{A}$ that contains A and B "amalgamated" over C.*
4. *$\mathcal{A}$ has existential closure (EC) if for every quantifier-free formula $\varphi(\overline{x}, \overline{g})$ with parameters from some $G \in \mathcal{A}$, and which is solved in some $\overline{G} \geq G$, there is some $H \in \mathcal{A}$ with $G \leq H$ and $H \models \exists \overline{x} \varphi(\overline{x}, \overline{g})$.*

Then there is a unique countable ω-homogeneous existentially closed group M with $Sk(M) = \mathcal{A}$.

The complexity of constructing existentially closed groups

Theorem (S.)

The existence of an existentially closed group is exactly at the level of $0'$.

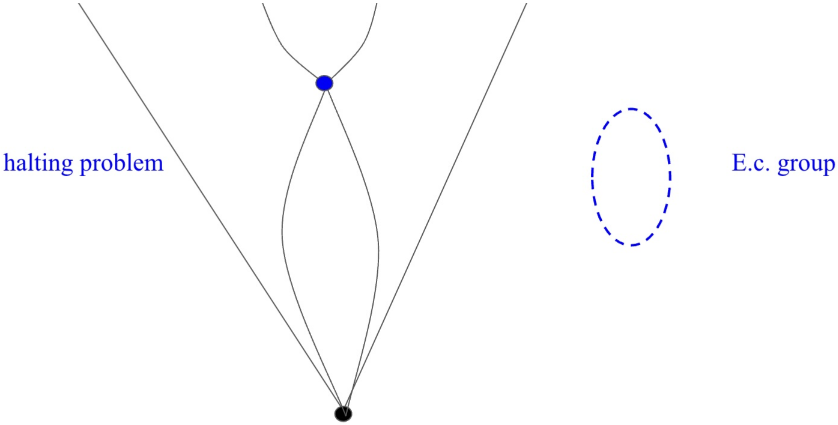
Every ec group computes $0'$.

Let M be an existentially closed group and $F = \langle \overline{f} \mid \overline{r} \rangle$ be a finitely presented group whose word problem computes $0'$. Define

$$\varphi_w(\overline{x}) = \bigwedge_i r(\overline{x}) = 1 \wedge w(\overline{x}) \neq 1$$

Note that $\varphi_w(\overline{x})$ is satisfiable iff $w \notin W(F)$.
Then $W(F)$ is c.e., so M-c.e. On the other hand, $\overline{W(H)}$ is M-c.e. by listing the tuples of elements of M and checking if they satisfy φ_w. □

The picture so far



Second question

Question

Can one characterise the degrees which compute word problems of finitely generated subgroups of every existentially closed group?

Second answer: the complexity of the subgroups of an e.c. group

Theorem ([Mil79], [Mac72b], [Zie80], S.)

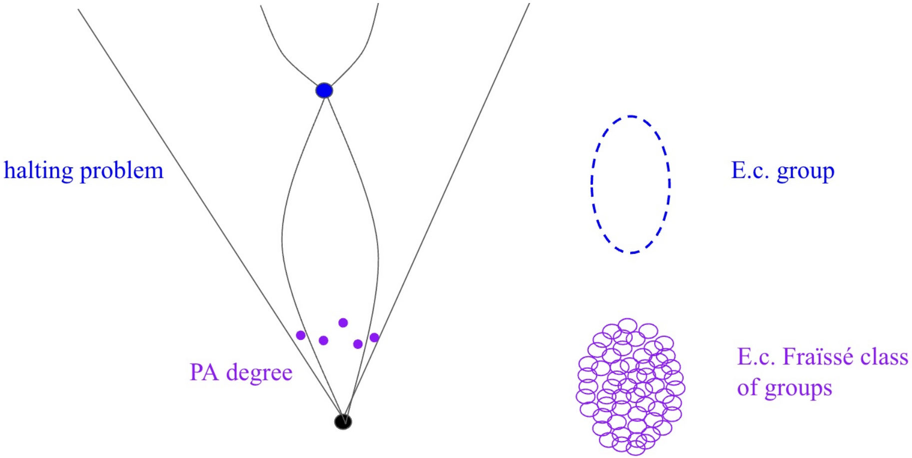
X computes the word problem of some finitely generated group in every existentially closed group iff X is a PA degree.

Proof Sketch that PA is enough.

Let $\mathcal{S}$ be a Scott set computed by X; i.e., a collection of X-computable sets that is closed under computability, joins, and "being PA in". Define $\mathcal{A}_{\mathcal{S}} = \{G \mid G \text{ is finitely generated and } W(G) \in \mathcal{S}\}$. This is an e.c. Fraïssé class.

□

The picture so far



Relatively atomic e.c. groups

Theorem ([Mil79], [Mac72b], [Zie80], S.)

X computes the word problem of every finitely generated group in some existentially closed group iff X is a PA degree.

Corollary

There are two existentially closed groups such that the only finitely generated subgroups which embed into both have solvable word problem.

Recall: A finitely generated group has solvable word problem iff its quantifier-free diagram is $\exists_1$-isolated.

Theorem (S.)

Let $A >_T 0'$ be $0'$-ce. Then A computes two existentially closed groups such that the only finitely generated subgroups which embed into both have solvable word problem.

Relatively atomic e.c. groups, continued

Theorem (S.)

Let $A >_T 0'$ be $0'$-ce. Then A computes two existentially closed groups such that the only quantifier-free types realised in both are $\exists_1$-isolated.

Proof sketch.

Fix $0'$-computable N. Let C be a countable set of constant symbols. We construct a theory T^M by induction satisfying the following:

- Group: "The structure determined by T^M models the axioms of group theory." Group: "The structure determined by T^M models the axioms of group theory."
- EC_m: "If $\exists \overline{x} \varphi_m(\overline{x})$ is consistent with T^M, then $T^M \models \varphi_m(\overline{c})$ for some set of constants $\overline{c}$." EC_m: "If $\exists \overline{x} \varphi_m(\overline{x})$ is consistent with T^M, then $T^M \models \varphi_m(\overline{c})$ for some set of constants $\overline{c}$."
- $R_{\langle \overline{c}, \overline{d} \rangle}$: "If $\mathrm{qftp}^M(\overline{c}) = \mathrm{qftp}^N(\overline{d})$, then this type is $\exists_1$-isolated."

□

Relatively atomic e.c. groups, continued

Theorem (S.)

Let $A >_T 0'$ be $0'$-ce. Then A computes two existentially closed groups such that the only quantifier-free types realised in both are $\exists_1$-isolated.

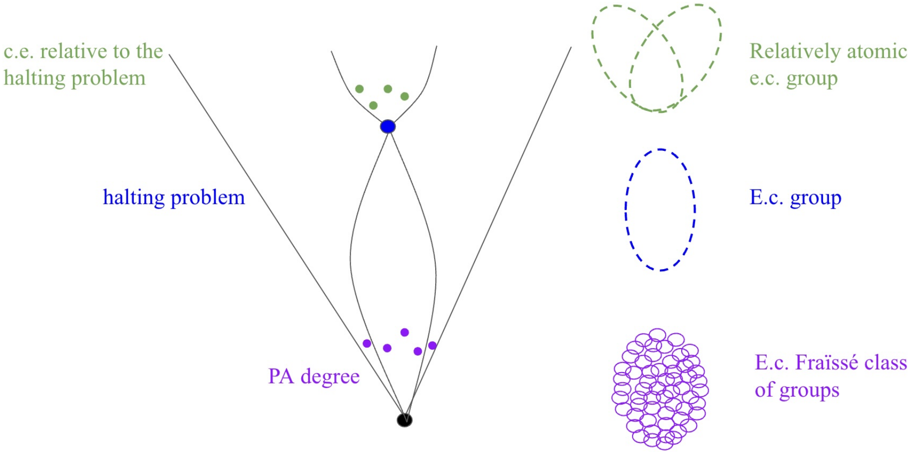
Proof sketch.

- $R_{\langle \overline{c}, \overline{d} \rangle}$: "If $\mathrm{qftp}^M(\overline{c}) = \mathrm{qftp}^N(\overline{d})$, then this type is $\exists_1$-isolated."

Stage s: Look for a "splittings" $\theta(\overline{c})$ such that θ and $\neg\theta$ are consistent with $T \upharpoonright r$ (where r is the max over higher priority restraints). If any splittings are found, request permission for them.

Satisfy highest priority requirement that gets permission at this stage, and reset lower priority requirements. □

The picture so far



Third question: Relativisation of Theorem 1

Theorem (S.)

The existence of an existentially closed group is equivalent to $0'$.

Question

Does the theorem above relativise?

Remark

We already know it does not relativise in the obvious way: the Fraïssé limit of $\mathcal{A}_0$, the class of fg groups with computable presentations, is $0'$-computable, but Craig's trick implies the existence of a computably presentable, $0'$-computable fg group.

Third answer

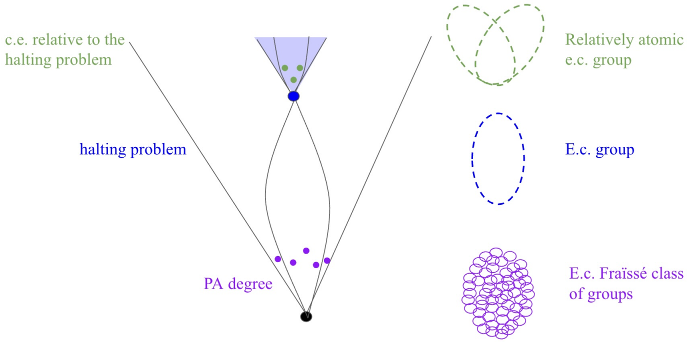
Definition

- $A \leq_e B$ if "positive information about A can be determined from positive information about B"; i.e., there is a c.e. W_e such that $n \in A \quad \text{if and only if} \quad \exists u(\langle n, u \rangle \in W_e \wedge D_u \subseteq B)$
- $\Gamma_e(B) := \{n \mid \exists u(\langle n, u \rangle \in W_e \wedge D_u \subseteq B)\}$
- $K_B := \bigoplus_{e \in \omega} \Gamma_e(B)$
- The *enumeration jump* of B is given by $J_e(B) := K_B \oplus \overline{K_B}$

Theorem (S.)

The existence of an existentially closed group containing a given finitely generated group G is equivalent to $J_e(G)$.

The proof involves Soskov's jump inversion theorem for $\mathcal{D}_e$ ([Sos00])!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Relatively atomic
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E.c. group
E.c. Fraïssé class
of groups

Thank you!

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