Pinned Distance Sets Using Effective Dimension

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Effective dimension

Fix a (universal) Turing machine U. Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The **Kolmogorov complexity of** x **at precision** r is

$$K_r(x) = \text{minimum length input } \pi \in \{0,1\}^* \text{ such that } U(\pi) = x{\upharpoonright}r,$$

where $x{\upharpoonright}r$ is the first r bits in the binary representation of x.

Let $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}.$$

The *(effective) packing dimension of* x is

$$\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}.$$

Dimension of distances

Problem: Let $x, y \in \mathbb{R}^2$. Give a lower bound on $\dim(|x-y|)$ depending on $\dim(x)$, $\dim(y)$ and $\dim(x \mid y)$.

- It is not hard to show that, if $\dim(x) = 2$, and $\dim^x(y) = 2$, then $\dim(|x-y|) = 1$.
  - Given x as an oracle, and given a 2^{-r} approximation of $|x-y|$, it takes at most r bits to describe y
  - $K_r^x(y) \leq K_r(|x-y|) + r + O(\log r)$, i.e., $K_r(|x-y|) \gtrsim r - O(\log r)$.
- A much harder question: Suppose $\dim(x), \dim(y) > 1$ and x, y are independent (share no information). Is it true that $\dim(|x-y|) = 1$?
- Any progress on this question, by the point-to-set principle, would lead to progress on Falconer's distance set problem, an important open question in geometric measure theory.

The Point-to-Set Principle

Theorem (J. Lutz and N. Lutz)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)$$
$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x).$$

- The Hausdorff dimension of a *set* is characterized by the (effective) dimension of the *points* in the set.
- Allows us to use computability to attack problems in geometric measure theory.

Distance sets

Let $E \subseteq \mathbb{R}^n$. The distance set of E is

$$\Delta E = \{|x - y| \mid x, y \in E\}.$$

More generally, if $x \in \mathbb{R}^n$, the pinned distance of E w.r.t. x is

$$\Delta_x E = \{|x - y| \mid y \in E\}.$$

Problem: Give a lower bound on the sizes of ΔE and $\Delta_x E$ in terms of the size of E.

Distance sets

When E is a finite set, Erdös conjectured that $|\Delta E|$ is at least (almost) linear in terms of $|E|$.

- In a breakthrough paper, Guth and Katz proved this in the plane.
- Still an important open problem for $\mathbb{R}^n$ with $n \geq 3$.

Falconer posed an analogous question for the case that E is infinite, known as Falconer's *distance set problem*.

- If $E \subseteq \mathbb{R}^n$ has $\dim_H(E) > n/2$, then ΔE has positive measure.
- Still open in all dimensions.
- Guth, Iosevich, Ou and Wang, proved that if $E \subseteq \mathbb{R}^2$ and $\dim_H(E) > 5/4$, then $\mu(\Delta E) > 0$.

Distance sets

Substantial progress has been made in a slightly different direction, on the Hausdorff dimension of *pinned distance sets* in the plane. We always assume E is analytic.

- Shmerkin proved that, if $\dim_H(E) > 1$ and $\dim_H(E) = \dim_P(E)$, then $\sup_{x\in E} \dim_H(\Delta_x E) = 1$.
- Liu showed that, if $\dim_H(E) = s \in (1, 5/4)$, then $\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{4}{3}s - \frac{2}{3}$.
- Shmerkin improved this bound when $\dim_H(E) = s \in (1, 1.04)$, by proving that
$$\sup_{x\in E} \dim_H(\Delta_x E) \geq 2/3 + 1/42 \approx 0.6904$$
- S. proved that
$$\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{\dim_H(E)}{4} + \frac{1}{2} > \frac{3}{4}$$

Our results

Theorem (Fiedler, S.)

Let $X, Y \subseteq \mathbb{R}^2$ such that Y is analytic, with $\dim_H(Y), \dim_H(X) > 1$. Then,

$$\sup_{x\in X} \dim_H(\Delta_x Y) \geq d\left(1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)}\right),$$

where $d = \min\{\dim_H(X), \dim_H(Y)\}$ and $D = \max\{\dim_P(X), \dim_P(Y)\}$. Moreover, if $D < \frac{d(3+\sqrt{5})-1-\sqrt{5}}{2}$, then $\dim_H(\Delta_x Y) = 1$.

Key points:

- Gives a lower bound on the pinned distance sets *based on both the Hausdorff and packing dimension of the set*.
- Strengthens the regularity result of Shmerkin: The packing dimension only needs to be sufficiently close to the Hausdorff dimension to prove the distance set conjecture.

Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Suppose that $x, y \in \mathbb{R}^2$, $e = \frac{y-x}{|y-x|}$ *satisfy the following.*

(C1) $\dim(x), \dim(y) > 1$

(C2) $K_r^x(e) \approx r$ *for all* r.

(C3) $K_r^x(y) \approx K_r(y)$ *for all sufficiently large* r.

(C4) $K_r(e \mid y) \approx r$ *for all* r.

Then $\dim^x(|x-y|) \geq d\left(1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)}\right)$, *where* $d = \min\{\dim(x), \dim(y)\}$ *and* $D = \max\{\mathrm{Dim}(x), \mathrm{Dim}(y)\}$.

Most of the work is in proving this theorem. We then use the point-to-set principle to conclude the classical theorem on the Hausdorff dimension of pinned distance sets.

Summarizing our results

Let $E \subseteq \mathbb{R}^2$ be analytic and $1 < d < \dim_H(E)$.

- $\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{d(d-4)}{d-5}$
  - This improves the best known bounds when $\dim_H(E) \in (1, 1.127)$.
- $\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{\dim_P(E)+1}{2\dim_P(E)}$.
- If $\dim_P(E) < \frac{d(3+\sqrt{5})-1-\sqrt{5}}{2}$, then $\sup_{x\in E} \dim_H(\Delta_x E) = 1$.
- There is a point $x \in E$ such that

$$\dim_P(\Delta_x E) \geq \frac{12-\sqrt{2}}{8\sqrt{2}} \approx 0.9356$$

  - Improves (slightly) the Keleti-Shermkin bound for packing dimension of pinned distance sets.

Regularity results

Theorem (Fiedler, S.)

Let $Y \subseteq \mathbb{R}^2$ *be analytic with* $\dim_H(Y) > 1$ *and* $\dim_P(Y) < 2\dim_H(Y) - 1$*. Let* $X \subseteq \mathbb{R}^2$ *be any set such that* $\dim_H(X) > 1$*. Then for all* $x \in X$ *outside a set of (Hausdorff) dimension one,*

$$\dim_H(\Delta_x Y) = 1.$$

Theorem (Fiedler, S.)

Let $Y \subseteq \mathbb{R}^2$ *be analytic with* $\dim_H(Y) > 1$*. Let* $X \subseteq \mathbb{R}^2$ *be any set such that* $\dim_H(X) = \dim_P(X) > 1$*. Then there is a subset* $F \subseteq X$ *such that,*

$$\dim_H(\Delta_x Y) = 1,$$

for all $x \in F$*. Moreover,* $\dim_H(X \setminus F) < \dim_H(X)$*.*

The End

Thank you!