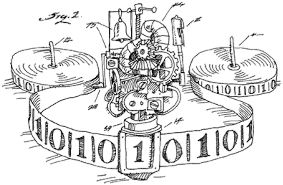
Machines running on random tapes and the probabilities of events



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joint work with Cenzer/Porter and Lewis-Pye

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Run a universal Turing machine on an arbitrary tape X.

What is the probability that it will

- halt? compute a total function?
- enumerate a computable set? enumerate a co-finite set?
- enumerate a set which computes the halting problem?
- compute an (in)computable function?
- halt with an output inside a certain set $A \neq \emptyset$?

These are reals in $(0, 1)$.

Becher et.al. showed that some of these are (highly) random.

Can we characterize them in terms of algorithmic randomness?

References

- Becher/Grigorieff. Random reals and possibly infinite computations part I: Randomness in $\emptyset'$. JSL 2005.
- Sureson. Random reals as measures of natural open sets. TCS 2005
- Becher/Figueira/Grigorieff/Miller. Randomness and halting probabilities. JSL 2006.
- Becher/Grigorieff. Random reals à la Chaitin with or without prefix-freeness. TCS 2007.

Universal halting probabilities

Shown to be exactly the 1-random left-c.e. reals in $(0, 1)$ by

- Chaitin (1975) – Solovay (1975)
- Calude/Hertling/Khousainov/Wang (2001)
- Kučera/Slaman (2001)

The Ω analysis.

For any Y let Ω^{Y} denote a Y-left-c.e. Y-random real in $(0, 1)$.

And let $1 - \Omega^{\mathrm{Y}}$ denote a Y-right-c.e. Y-random real in $(0, 1)$.

Can we characterize all natural universal probabilities in terms of relativized Ω numbers?

Characterization of probabilities I

Totality	$1 - \Omega^{\emptyset'}$
Enumeration of a computable set	$\Omega^{\emptyset^{(2)}}$
Enumeration of a co-finite set	$\Omega^{\emptyset^{(2)}}$
Enumeration of a set which computes $\emptyset'$	$\Omega^{\emptyset^{(3)}}$
Universality probability	$1 - \Omega^{\emptyset^{(3)}}$

- Barmpalias/Cenzer/Porter TCS (2017)
- Barmpalias/Dowe Phi. Trans. R. Soc. (2012)

What about

- computing a computable function?
- computing a co-finite set?

These questions are not subject to the previous analysis.

Indeed these probabilities are do not need to be random.

However the analysis is based on:

- recent and not-so-recent properties of omega numbers;
- some theory of lowness for randomness;
- additional constructions of universal machines.

Characterization of probabilities II

Computing incomputable set	$1 - \Omega^{\emptyset'}$
Computing a computable set	$\emptyset'$-d.c.e. reals in $(0, 1)$
Computing cofinite set	$\emptyset'$-d.c.e. reals in $(0, 1)$

Barmpalias/Cenzer/Porter Arxiv 1612.08537 (2017)

Computing an (in)computable set

Why the difference of two $\emptyset'$-left-c.e. reals?

Given machine M:

- TOT(M) is a Π^0_2 class
- INCTOT(M) is a Π^0_3 class.

Let (V_i) be a universal Martin-Löf test and let:

$$\mathrm{INCTOT}^*(M) = \mathrm{TOT}(M) \cap \{X \mid X \in \cap_i V_i^{M(X)}\}.$$

For every 2-random X we have

$$X \in \mathrm{INCTOT}(M) \Leftrightarrow X \in \mathrm{INCTOT}^*(M).$$

...by the theory of lowness for randomness.

Computing an (in)computable set

Hence

$$\mu\,(\text{INCTOT}(\text{M})) = \mu\,(\text{INCTOT}(\text{M})^*)\,.$$

Also $\text{INCTOT}(\text{M})^*$ is a Π^0_2 class.

So

$$\mu\,(\text{TOT}(\text{M}) - \text{INCTOT}(\text{M})^*)$$

is a $\emptyset'$-d.c.e. real.

The other direction relies on a recent fact about Ω numbers.

The Ω derivation theorem.

Given a left-c.e. approximation $(\alpha_s) \to \alpha$ and $(\Omega_s) \to \Omega$,

$$\lim_s \frac{\alpha - \alpha_s}{\Omega - \Omega_s} = r \in [0, \infty)$$

$r \neq 0$	$\iff$	α	is 1-random
$r \neq 1$	$\iff$	$\alpha - \Omega$	is 1-random.

If α is 1-random then

$r \in (0, 1)$	$\iff$	$\alpha - \Omega$ is left-c.e.
$r > 1$	$\iff$	$\alpha - \Omega$ is right-c.e.
$r = 1$	$\iff$	$\alpha - \Omega$ is properly d.c.e.

Barmpalias/Lewis Arxiv 1604.00216 (2016)

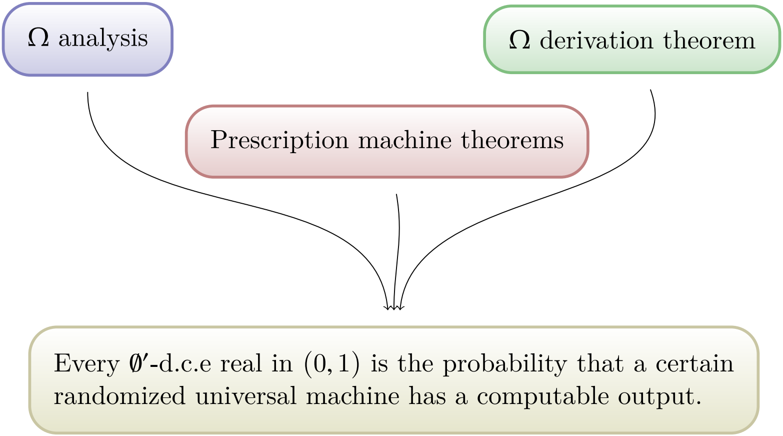
Prescription machine theorems

Given a Σ^0_2 prefix-free set of strings Q, there exist machines M_0, M_1 such that

- $M_0(X)$ is computable iff $X \in [\![Q]\!]$
- $M_1(X)$ is computable iff $X \notin [\![Q]\!]$

for every Martin-Löf random real X.

The harder direction



Restricted halting probability

Given the universal prefix-free machine U and a set X let

$$\Omega(\mathrm{X}) := \sum_{\mathrm{U}(\sigma)\downarrow\in \mathrm{X}} 2^{-|\sigma|}$$

the probability that U halts with output in X.

Grigorieff (2002) asked if the arithmetical complexity of X is reflected on the randomness of $\Omega_{\mathrm{U}}(\mathrm{X})$.

Becher/Figueira/Grigorieff/Miller (2006) showed that

- $\Omega_U(X)$ is rational for some $X \leq_T \emptyset'$;
- $\Omega_U(X)$ is 1-random for Σ^0_n-complete X;
- $\Omega_U(X)$ is not n-random for $X \in \Sigma^0_n$, n>1;

...giving a negative answer to Grigorieff's question.

If $X \neq \emptyset$ is Π^0_1 then is $\Omega_U(X)$ Martin-Löf random?

This question was discussed and/or attempted in

- Becher/Grigorieff. Random reals and possibly infinite computations part I: Randomness in $\emptyset'$. JSL 2005.
- Becher/Figueira/Grigorieff/Miller. Randomness and halting probabilities. JSL 2006.
- Figueira/Stephan/Wu. Randomness and universal machines. J. Complexity 2006.
- Miller/Nies. Randomness and computability: open questions. Bul. Symb. Logic 2006.

Overview of the argument

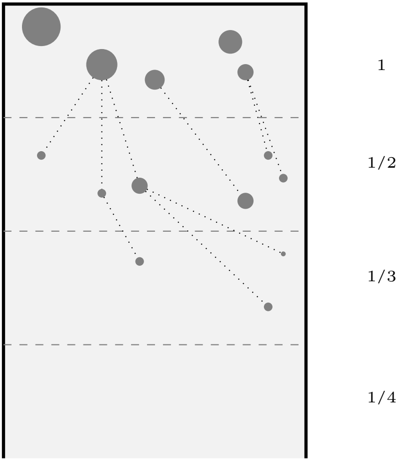
If X is a Π^0_1 set and $\Omega_U(X)$ is a right-c.e. real then $\Omega_U(X)$ is not Martin-Löf random.

Ω derivation theorem

Adding a random left-c.e. real to a non-random d.c.e. real gives a random c.e. real.

If X is a nonempty Π^0_1 set, the number $\Omega_U(X)$ is a Martin-Löf random left-c.e. real.

Decanter argument



Thanks! – and main references

- Barmpalias/Lewis. Differences of halting probabilities.
  Arxiv: 1604.00216 (2016)

- Barmpalias/Cenzer/Porter The probability of a computable output from a random oracle. Arxiv:1612.08537 (2017)

- Barmpalias/Cenzer/Porter Random numbers as probabilities of machine behaviour. TCS (2017)