Computable numberigs

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Non-clasical computable numberings(Goncharov-Sorbi)

Let S^* be a set of objects and L be some formal language. We can take some interpretation Int from L onto S^*.

A numberings ν from N onto S where $S \subseteq S^*$ is computable if there is a computable function from N to L such that $\nu(n) = int(f(n))$ for any n.

$\nu \leq \mu$

$R(S, int)$ Rogers semilattice

Atithmetical and Hyperarithmetical hierarhies. (Σ^0_α)

S.Badaev-S.Goncharov-A.Sorbi and S.Podzorov.

Open problem. Is there a family c.e. sets with exactly 2 (n+2) minimal computable numberings? (Yu.Ershov)

Open problem. Is there a family S such that $R(S, \Sigma^0_{n+2})$ is not isomorphic to $R(S_0, \Sigma^0_{n+1})$ for any Σ_{n+1}-computable family S_0.

Open problem. Let S, S' be finite families. In what cases $R(S, \Sigma^0_1)$ and $R(S', \Sigma^0_1)$ are isomorphic?

Arithmetical sets

Theorem

(D. Velegzhanina) There is a fridberg arithmetical computable numbering of all arithmetical sets.

Computable functionals(Yu. Ershov)

Theorem

(S.Ospichev) There is a fridberg computable numbering of all τ-computable functionals sets.

Analitical hierarhy, James C. OWINGS, JR

Theorem

There is not a meta-c.e. numbering S(α)($\alpha < \omega_1$) Π^1_1-sets such that it is friedberg numbering of all Π^1_1-sets.

Theorem

There is not a Σ^1_1-numbering Σ^1_1-sets such that it is a friedberg numbering of all Σ^1_1-sets for $1 \leq n \leq 2$.

Analytical hierarchy, M. Dorzhieva

Theorem

There is not a Σ^n_1-numbering S(α)($\alpha < \omega_1$) Π^1_1-sets such that it is friedberg numbering of all Σ^n_1-sets for $1 \leq n \leq 2$.

Proof is without metarecursion.

Open problem. Is it true that there is not a Σ^3_1-numbering S(α)($\alpha < \omega_1$) Π^1_1-sets such that it is friedberg numbering of all Π^3_1-sets.

Ershov hierarchy

Goncharov-Lempp-Solomon, Badaev-Lempp, Ospichev, Talasbaeva and so on.

Open problem. Is there of a family of Σ_2^{-1}-множеств с точно двумя Σ_2^{-1}-computable numberings.

Strongly constructive models = decidable models

A.Malcev gave a notion of constructive model and suggested some main questions.

Yu.Ershov in 1968 has introduced the notion of strongly constructive model.

M. Morley in 1971 has introduced an equivalent notion of decidable model.

The definitions of strong constructive model and decidable model are equivalent. This notions play important role Computable Model Theory.

some basic problems of computable models

We consider the following two basic problems:

- The existence of computable (decidable) models with some properties.
- The dimension of computable models relative to some types of equivalence and presentations.

We consider the following basic class of questions on the base of these problems:

- To describe computable structures with decidable representations from some definable classes.
- To describe computable autostable relative to same equivalence models from some definable classes.
- To describe computable models with finite (fix) dimension.
- ....

Computable Characterization, R. Shore question

Universal numberings for classes (A.Nurtazin).

Goal:

To study the relations between definability of different classes of computable structures and their algorithmic complexity.

Index sets

Goncharov and Knight

Suppose that K is a class of computable structures of a signature σ. Suppose also that K is closed under isomorphism. The **index set** of the class K is the set

$$I(K) = \left\{ e \in \omega : \exists \mathfrak{M} \in K \left(\varphi_e = \chi_{D(\mathfrak{M})} \right) \right\},$$

where $\chi_{D(\mathfrak{M})}$ is the characteristic function of the atomic diagram of $\mathfrak{M}$.

Main problem

Problem

Obtain an estimate for the complexity of index sets for different familiar classes of structures.

Index sets for decidable structures

Theorem (Fokina, 2007, 2009)

(1) The index set of decidable structures of a signature σ is an m-complete Σ^0_3 set.

(2) The index set of decidable countably categorical structures of σ is m-complete $(\Sigma^0_3 - \Sigma^0_3)$ (where $\Sigma^0_3 - \Sigma^0_3$ denotes the difference of two Σ^0_3 sets).

(3) The index set of computable structures of σ with decidable theories is m-complete $\Sigma^0_{\omega+1}$.

Good News!!! MATTHEW HARRISON-TRAINOR. "THERE IS NO CLASSIFICATION OF THE DECIDABLY PRESENTABLE STRUCTURES".

Index set of autostable structures

Theorem (Downey, Kach, Lempp, Lewis-Pye, Montalbán, Turetsky, 2015)

The index set of autostable structures of a signature σ is m-complete Π^1_1.

Suppose that S is an m-complete Π^1_1 set. Downey et al. built a computable sequence $\{\mathfrak{A}_n\}_{n\in\omega}$ of directed graphs such that

- if $n \in S$, then $\mathfrak{A}_n$ is autostable;
- if $n \notin S$, then the hyperarithmetical dimension of $\mathfrak{A}_n$ is infinite, i. e. there are computable structures $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots$ such that $\mathfrak{B}_i \cong \mathfrak{A}_n$ and $\mathfrak{B}_i \not\cong_{\mathrm{HYP}} \mathfrak{B}_j$ for $i \neq j$.

Finite computable dimension

The **computable dimension** of a computable structure $\mathfrak{M}$ is the number of computable copies of $\mathfrak{M}$ up to computable isomorphism.

Theorem (Goncharov, 1980)

Let n be a natural number such that $n \geq 2$. There exists a computable structure $\mathfrak{M}$ with computable dimension n.

Theorem (Bazhenov, Goncharov, Marchuk and ..)

Suppose that $2 \leq n < \omega$. The index set of computable structures of a signature σ with computable dimension n is m-complete Π^1_1.

Autostability relative to strong constructivizations

A computable structure $\mathfrak{M}$ is **autostable relative to strong constructivizations** if $\mathfrak{M}$ is strongly constructivizable and, for any decidable copies $\mathfrak{N}_0$ and $\mathfrak{N}_1$ of $\mathfrak{M}$, there exists a computable isomorphism from $\mathfrak{N}_0$ onto $\mathfrak{N}_1$.

For the sake of brevity, we will call autostable relative to strong constructivizations structures SC-**autostable** structures.

Index set of strongly computable almost prime structures

Let σ be a signature. We define the index set
$\mathrm{SC}-\mathrm{almostPrime}_\sigma$
$= \{e \in \omega : \mathfrak{M}_e$ is strongly constructivizable structure$\}$.

Theorem 1 (Goncharov, 2014)

Suppose that σ is a nontrivial computable signature. Then the index set $\mathrm{SC}-\mathrm{almostPrime}_\sigma$ of structures of σ is m-complete $\Sigma^0_{\omega+2}$.

Autostable strongly constructivizable structures

The main focus of this talk is autostability relative to strong constructivizations.

A structure $\mathfrak{M}$ is **strongly constructivizable** if $\mathfrak{M}$ has a decidable copy.

Theorem (Bazhenov, Goncharov, Marchuk)

The index set DecAut_σ of strongly constructivizable autostable structures of a signature σ is m-complete Σ^0_3.

SC-autostability for linear orderings

LO denotes the class of all linear orderings.

Theorem 2 (Bazhenov, Goncharov, Marchuk)

The index set $\mathrm{SCAut}(LO)$ (i. e. the index set of SC-autostable linear orderings) is m-complete $\Sigma^0_{\omega+2}$.

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Theorem 2 (Bazhenov, Goncharov, Marchuk)

The index set $\mathrm{SCAut}(LO)$ (i. e. the index set of SC-autostable linear orderings) is m-complete $\Sigma^0_{\omega+2}$.

Corollary

Let PO be the class of partial orderings, and let DL be the class of distributive lattices. The index sets $\mathrm{SCAut}(PO)$ and $\mathrm{SCAut}(DL)$ are m-complete $\Sigma^0_{\omega+2}$.

Symmetric irreflexive graphs

Using the coding of directed graphs into symmetric irreflexive graphs (Hirschfeldt, Khoussainov, Shore, Slinko, 2002), we show the following.

Proposition (Goncharov, Marchuk)

Let UG be the class of symmetric irreflexive graphs. The index set $\mathrm{SCAut}(UG)$ is m-complete $\Sigma^0_{\omega+2}$.

Structures with two equivalences

Let $2EQ$ denote the class of structures with two equivalences.

Marchuk constructed the effective transformation of linear orderings into structures with two equivalences.

Theorem (Marchuk)

The index set $\mathrm{SCAut}(2EQ)$ is m-complete $\Sigma^0_{\omega+2}$.

SC-autostability for Boolean algebras

Let BA denote the class of Boolean algebras.

Theorem 3 (Bazhenov, Goncharov, Marchuk)

The index set $\mathrm{SCAut}(BA)$ is m-complete $\Sigma^0_{\omega+2}$.

Corollary

Suppose that R is the class of commutative associative rings, and SG is the class of commutative semigroups. The index sets $\mathrm{SCAut}(R)$ and $\mathrm{SCAut}(SG)$ are m-complete $\Sigma^0_{\omega+2}$.

Summary of the results

The index set $\mathrm{SCAut}(K)$ is an m-complete $\Sigma^0_{\omega+2}$ set for the following classes K:

- linear orderings,
- Boolean algebras,
- symmetric irreflexive graphs,
- partial orderings,
- distributive lattices,
- rings,
- commutative semigroups,

Decidable groups

Theorem (Bazhenov, Goncharov, Marchuk, 2016)

Let $\mathcal{G}$ be the class of all groups. The exact complexity of the index set $\mathrm{SCAut}(\mathcal{G})$ is a $\Sigma^0_{\omega+2}$ set.

Characterization of complexity in case of bounded decidability.

Theorem (Goncharov)

The index set of 2-decidable, computably categorical structures is Π^0_3 complete.

Theorem (Fokina, Goncharov, Harizanov, Kudinov, Turetsky, 2015)

The index set of 1-decidable, computably categorical structures is Π^0_4 complete.

Theorem (Fokina, Goncharov, Harizanov, Kudinov, Turetsky, 2015)

For any $n \geq 2$ and $m \leq n-2$, the index set of n-decidable, categorical relative to m-decidable presentations structures is Σ^0_3 complete.

Autostability spectra

Definition (2009)

The **autostability spectrum** of a computable structure $\mathfrak{M}$ is the set

$$\mathrm{AutSpec}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is } \mathbf{d}\text{-autostable}\}.$$

A Turing degree $\mathbf{d}_0$ is the **degree of autostability** of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the set $\mathrm{AutSpec}(\mathfrak{M})$.

Autostability spectra and degrees of autostability are also called **categoricity spectra** and **degrees of categoricity** respectively.

Looking even further: SC-autostability spectra

Let $\mathbf{d}$ be a Turing degree. A strongly constructivizable structure $\mathfrak{M}$ is **$\mathbf{d}$-autostable relative to strong constructivizations** (**$\mathbf{d}$-SC-autostable**) if for any decidable copies $\mathfrak{N}_0$ and $\mathfrak{N}_1$ of $\mathfrak{M}$, there exists a $\mathbf{d}$-computable isomorphism $f\colon \mathfrak{N}_0 \to \mathfrak{N}_1$.

Definition (Goncharov, 2011)

The **autostability spectrum relative to strong constructivizations** (**SC-autostability spectrum**) of a strongly constructivizable structure $\mathfrak{M}$ is the set

$$\mathrm{AutSpec}_{SC}(\mathfrak{M}) = \{\mathbf{d} : \mathfrak{M} \text{ is } \mathbf{d}\text{-}SC\text{-autostable}\}.$$

Degrees of SC-autostabilty

A Turing degree $\mathbf{d}_0$ is the **degree of autostability relative to strong constructivizations** (**degree of SC-autostability**) of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{SC}(\mathfrak{M})$.

Observation

Every degree of SC-autostability is a degree of autostability.

Degrees of SC-autostabilty

A Turing degree $\mathbf{d}_0$ is the **degree of autostability relative to strong constructivizations (degree of SC-autostability)** of $\mathfrak{M}$ if $\mathbf{d}_0$ is the least degree in the spectrum $\mathrm{AutSpec}_{SC}(\mathfrak{M})$.

Observation

Every degree of SC-autostability is a degree of autostability.

Open problem

Is the converse true?

Degrees of autostability for ordinals

We extend the result of Ash to the following.

Theorem (Bazhenov)

Suppose that k is a non-zero natural number, and γ is a computable infinite ordinal. Let α be an ordinal.

1. Suppose that $\omega^k \leq \alpha < \omega^{k+1}$. Then $\mathbf{0}^{(2k-1)}$ is the degree of autostability of α.
2. Suppose that $\omega^\gamma \leq \alpha < \omega^{\gamma+1}$. Then $\mathbf{0}^{(2\gamma)}$ is the degree of autostability of α.

SC-autostability and degrees of autostability

Corollary

Suppose that $\beta \leq \omega$. There exists a SC-autostable linear ordering with the degree of autostability $\mathbf{0}^{(\beta)}$.

SC-autostability and degrees of autostability

Corollary

Suppose that $\beta \leq \omega$. There exists a SC-autostable linear ordering with the degree of autostability $\mathbf{0}^{(\beta)}$.

We also obtain a similar result for Boolean algebras.

Theorem 4 (Bazhenov, Marchuk)

Let $\mathfrak{B}$ be a SC-autostable Boolean algebra. Then $\mathfrak{B}$ has the degree of autostability.

The proof provides the exact calculations of the degrees. Therefore, Theorem 4 yields the following.

Corollary

Suppose that $\beta \leq \omega$. There exists a SC-autostable Boolean algebra with the degree of autostability $\mathbf{0}^{(\beta)}$.

Examples of degrees of autostability

Autostability	SC-autostability
$\mathbf{0}^{(\alpha)}$, $\alpha < \omega_1^{CK}$ (Fokina, Kalimullin, Miller, 2010; Csima, Franklin, Shore, 2013)	$\mathbf{0}^{(\alpha)}$, $\alpha < \omega_1^{CK}$ (for Boolean algebras)
$\mathbf{d}$, d.c.e. (Fokina, Kalimullin, Miller, 2010)	$\mathbf{d}$, c.e. (Goncharov, 2011; for the structures of $\sigma = \{P_k^1 : k \in \omega\}$)
$\mathbf{d}$, d.c.e. over $\mathbf{0}^{(\alpha+1)}$ (Fokina, Kalimullin, Miller, 2010; Csima, Franklin, Shore, 2013)	$\mathbf{d}$, c.e. over $\mathbf{0}^{(\alpha+1)}$ (for the structures of $\sigma = \sigma_{BA} \cup \{P_k^1 : k \in \omega\}$)

σ_{BA} denotes the signature for Boolean algebras.

Degree of autostability and degree of SC-autostability

Remark

Let $\mathbf{d}$ be a Turing degree. If a decidable structure $\mathfrak{M}$ is $\mathbf{d}$-autostable, then $\mathfrak{M}$ is $\mathbf{d}$-SC-autostable.

Proposition (Bazhenov, Marchuk)

Suppose that $0 \leq \alpha \leq \beta \leq \omega$. There exists a decidable structure $\mathfrak{M}$ of a finite signature σ such that

- $\mathbf{0}^{(\alpha)}$ is the degree of SC-autostability of $\mathfrak{M}$, and
- $\mathbf{0}^{(\beta)}$ is the degree of autostability of $\mathfrak{M}$.

Structures with no degree of autostability

Theorem (Miller, 2009)

There exists a computable algebraic field F such that the following holds.

(a) F is not autostable.

(b) There are low degrees $\mathbf{c}$ and $\mathbf{d}$ from $\mathrm{AutSpec}(F)$ such that $\mathbf{c} \wedge \mathbf{d} = \mathbf{0}$.

In particular, F has no degree of autostability.

Structures with no degree of SC-autostability

σ_{BA} denotes the signature for Boolean algebras.

Theorem 5 (Bazhenov)

There exists a decidable structure $\mathfrak{M}$ of the signature $\sigma_{BA} \cup \{P_k^1 : k \in \omega\}$ with the following properties.

(a) $\mathfrak{M}$ is a prime model.

(b) $\mathfrak{M}$ is not SC-autostable.

(c) There are low degrees $\mathbf{c}$ and $\mathbf{d}$ from $\mathrm{AutSpec}_{SC}(\mathfrak{M})$ such that $\mathbf{c} \wedge \mathbf{d} = \mathbf{0}$.

(d) Every computable copy of $\mathfrak{M}$ is decidable.

In particular, $\mathrm{AutSpec}_{SC}(\mathfrak{M}) = \mathrm{AutSpec}(\mathfrak{M})$, and $\mathfrak{M}$ has no degree of SC-autostability.

Structures with no degree of SC-autostability

Proposition (Bazhenov, Marchuk)

Suppose that $1 \leq \beta \leq \omega$. There exists a decidable structure $\mathfrak{M}$ such that:

(a) $\mathfrak{M}$ has no degree of SC-autostability, and

(b) $\mathbf{0}^{(\beta)}$ is the degree of autostability of $\mathfrak{M}$.

Structures with no degree of SC-autostability

Proposition (Bazhenov, Marchuk)

Suppose that $1 \leq \beta \leq \omega$. There exists a decidable structure $\mathfrak{M}$ such that:

(a) $\mathfrak{M}$ has no degree of SC-autostability, and

(b) $\mathbf{0}^{(\beta)}$ is the degree of autostability of $\mathfrak{M}$.

Open problem

Does there exist a structure $\mathfrak{M}$ with degree of SC-autostability and with no degree of autostability?

Thanks for the invitation on this meeting.