Strong and non-strong degrees of categoricuty

Kalimullin I.Sh.

Kazan Federal University
e-mail:ikalimul@gmail.com

Dagstuhl Seminar, February 2017

Degrees of Categoricity

- A computable structure $\mathcal{A}$ has a degree of categorcity $\mathbf{a}$ if $\mathbf{a}$ is the least Turing degree such that for every computable compies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

Degrees of Categoricity

- A computable structure $\mathcal{A}$ has a degree of categorcity $\mathbf{a}$ if $\mathbf{a}$ is the least Turing degree such that for every computable compies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.
- A degree of categoricity $\mathbf{a}$ of a computable structure $\mathcal{A}$ is strong if there are computable compies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ such that $\mathbf{a} \leq_T f$ for every isomorphism f from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

Degrees of Categoricity

- A computable structure $\mathcal{A}$ has a degree of categorcity $\mathbf{a}$ if $\mathbf{a}$ is the least Turing degree such that for every computable compies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.
- A degree of categoricity $\mathbf{a}$ of a computable structure $\mathcal{A}$ is strong if there are computable compies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ such that $\mathbf{a} \leq_T f$ for every isomorphism f from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

Question. Does every degree of categoricity is strong?

Non-strong degrees of categoricity

- (Csima,Stephenson). There is a computable rigid structure of computable dimension 3 whose degree of categorcity is not strong.

Non-strong degrees of categoricity

- (Csima,Stephenson). There is a computable rigid structure of computable dimension $\mathbf{3}$ whose degree of categorcity is not strong.
- (Bazhenov, K, Yamaleev). There is a computable rigid structure with degree of categoricity $\mathbf{0}'$ and this degree is not strong.

Non-strong degrees of categoricity

- (Csima,Stephenson). There is a computable rigid structure of computable dimension $\mathbf{3}$ whose degree of categorcity is not strong.
- (Bazhenov, K, Yamaleev). There is a computable rigid structure with degree of categoricity $\mathbf{0}'$ and this degree is not strong.

Both examples have the property that there are two pairs of copies $\mathcal{A}_0^0, \mathcal{A}_0^1$ and $\mathcal{A}_1^0, \mathcal{A}_1^1$ such that the degree of categoricity is computable in $f_0 \oplus f_1$, where f_0 maps $\mathcal{A}_0^0$ onto $\mathcal{A}_0^1$ and f_1 maps $\mathcal{A}_1^0$ onto $\mathcal{A}_1^1$.

Categoricity spectra

- The categoricity spectrum of a computable structure $\mathcal{A}$ is a collection of all Turing degrees $\mathbf{a}$ such that for every computable copies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

Categoricity spectra

- The categoricity spectrum of a computable structure $\mathcal{A}$ is a collection of all Turing degrees $\mathbf{a}$ such that for every computable copies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.
- Or, equivalently

$$\mathbf{CatSp}(A) = \bigcap_{\mathcal{A}^0 \cong \mathcal{A}^1 \cong \mathcal{A}} \mathbf{IsSp}(\mathcal{A}^0, \mathcal{A}^1),$$

where $\mathbf{IsSp}(\mathcal{A}^0, \mathcal{A}^1)$ is the degrees computing isomorphisms from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

Categoricity spectra

- The categoricity spectrum of a computable structure $\mathcal{A}$ is a collection of all Turing degrees $\mathbf{a}$ such that for every computable copies $\mathcal{A}^0 \cong \mathcal{A}$ and $\mathcal{A}^1 \cong \mathcal{A}$ there is an $\mathbf{a}$-computable isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$.
- Or, equivalently

$$\mathbf{CatSp}(A) = \bigcap_{\mathcal{A}^0 \cong \mathcal{A}^1 \cong \mathcal{A}} \mathbf{IsSp}(\mathcal{A}^0, \mathcal{A}^1),$$

where $\mathbf{IsSp}(\mathcal{A}^0, \mathcal{A}^1)$ is the degrees computing isomorphisms from $\mathcal{A}^0$ onto $\mathcal{A}^1$.
- The degree of categoricity of $\mathcal{A}$ = the least element of $\mathbf{CatSp}(A)$.

Spectral dimension

- The spectral dimension $\mathbf{SpDim}(\mathcal{A})$ of a computable structure $\mathcal{A}$ is the least $n \leq \omega$ such that

$$\mathbf{CatSp}(A) = \bigcap_{i<n} \mathbf{IsSp}(\mathcal{A}_i^0, \mathcal{A}_i^1)$$

for some choice of computable copies $\mathcal{A}_i^0 \cong \mathcal{A}_i^1 \cong \mathcal{A}$, $i < n$.

Spectral dimension

- The spectral dimension $\mathbf{SpDim}(\mathcal{A})$ of a computable structure $\mathcal{A}$ is the least $n \leq \omega$ such that

$$\mathbf{CatSp}(A) = \bigcap_{i<n} \mathbf{IsSp}(\mathcal{A}_i^0, \mathcal{A}_i^1)$$

  for some choice of computable copies $\mathcal{A}_i^0 \cong \mathcal{A}_i^1 \cong \mathcal{A}$, $i < n$.
- If $\mathcal{A}$ has a degree of categoricity $\mathbf{a}$ then $\mathbf{SpDim}(\mathcal{A})$ is the least $n \leq \omega$ such that $\mathbf{a} \leq_T \bigoplus_{i<n} f_i$ for every $f_i \in \mathbf{IsSp}(\mathcal{A}_i^0, \mathcal{A}_i^1)$, $i < n$.

Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\mathbf{SpDim}(\mathcal{A}) = 0$.

Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\mathbf{SpDim}(\mathcal{A}) = 0$.
- $\mathbf{SpDim}(\mathcal{A}) \leq \mathbf{Dim}(\mathcal{A}) - 1$.

Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\mathbf{SpDim}(\mathcal{A}) = 0$.
- $\mathbf{SpDim}(\mathcal{A}) \leq \mathbf{Dim}(\mathcal{A}) - 1$.
- If $\mathbf{a}$ is a degree of categoricity then $\mathbf{SpDim}(\mathcal{A}) = 1$ iff $\mathbf{a}$ is strong.

Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\mathbf{SpDim}(\mathcal{A}) = 0$.
- $\mathbf{SpDim}(\mathcal{A}) \leq \mathbf{Dim}(\mathcal{A}) - 1$.
- If $\mathbf{a}$ is a degree of categoricity then $\mathbf{SpDim}(\mathcal{A}) = 1$ iff $\mathbf{a}$ is strong.
- For rigid structures $\mathbf{SpDim}(\mathcal{A}) < \omega$ iff $\mathcal{A}$ has a degree of categoricity.

Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\mathbf{SpDim}(\mathcal{A}) = 0$.
- $\mathbf{SpDim}(\mathcal{A}) \leq \mathbf{Dim}(\mathcal{A}) - 1$.
- If $\mathbf{a}$ is a degree of categoricity then $\mathbf{SpDim}(\mathcal{A}) = 1$ iff $\mathbf{a}$ is strong.
- For rigid structures $\mathbf{SpDim}(\mathcal{A}) < \omega$ iff $\mathcal{A}$ has a degree of categoricity.
- There are rigid structures $\mathcal{A}$ without degree of categoricity (Fokina, Frolov, K). For such structures we have $\mathbf{SpDim}(\mathcal{A}) = \omega$.

Main result

Theorem. (Bazhenov, K., Yamaleev). For every $n < \omega$ there is a rigid computable structure of degree of categoricity $\mathbf{0}'$ such that $\mathbf{SpDim}(\mathcal{A}) = n$.

Main result

Theorem. (Bazhenov, K., Yamaleev). For every $n < \omega$ there is a rigid computable structure of degree of categoricity $\mathbf{0}'$ such that $\mathbf{SpDim}(\mathcal{A}) = n$.

Open Question. Is there a computable structure having a degree of categoricity such that $\mathbf{SpDim}(\mathcal{A}) = \omega$?

The proof from the computability theory point of view

- For a (partial) function $R : \omega \times \omega \to \{0, 1\}$ and $G : \omega \to \omega$ we set
$$(R \diamond G)(x) = R(x, G(x))$$

The proof from the computability theory point of view

- For a (partial) function $R : \omega \times \omega \to \{0, 1\}$ and $G : \omega \to \omega$ we set
$$(R \diamond G)(x) = R(x, G(x))$$

- If G and $R \diamond G$ are total and R is partially computable then $R \diamond G \leq_T G$.

The proof from the computability theory point of view

- For a (partial) function $R : \omega \times \omega \to \{0, 1\}$ and $G : \omega \to \omega$ we set
$$(R \diamond G)(x) = R(x, G(x))$$

- If G and $R \diamond G$ are total and R is partially computable then $R \diamond G \leq_T G$.
- The proof consists of the construction of limitwise monotonic $G \equiv_T \emptyset'$ such that $G(x) < 2^n$ and
$$G \not\leq_T (R_1 \diamond G) \oplus \cdots \oplus (R_{n-1} \diamond G)$$
for every $\{0, 1\}$-valued partial computable $R_1, \ldots, R_{n-1}$.

THANK YOU FOR ATTENTION!