Computability, proof mining and metric regularity

(work in progress, partly with Genaro Lopéz-Acedo)

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Moduli of uniqueness

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Adapted to fixed point problems it reads as:

Definition

Let (X, d) be a metric space, and let $T : X \to X$. We say that

$$\phi : (0, \infty) \times (0, \infty) \to (0, \infty)$$

is a modulus of uniqueness (for fixed points of T) if for all $\varepsilon > 0$ and $x, y \in X$ such that $d(x, y) \leq d$ we have

$$\left.\begin{array}{r} d(x, Tx) < \phi(\varepsilon, d) \\ \\ d(y, Ty) < \phi(\varepsilon, d) \end{array}\right\} \Rightarrow d(x, y) < \varepsilon.$$

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In a normed setting one requires $\|x\|, \|y\| \leq d$.

Computational use of moduli of uniqueness

If T possesses arbitrarily good approximate fixed points and we have an algorithm which, given $\varepsilon > 0$, computes $p_\varepsilon \in X$ s.t. $d(p_\varepsilon, p_{\varepsilon'}) \leq d$ and

$$d(Tp_\varepsilon, p_\varepsilon) < \varepsilon.$$

Then $\{p_{\phi(2^{-n})}\}_n$ is a 2^{-n}-Cauchy sequence whose limit (for complete X and continuous T) is a fixed point of T.

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A modulus of uniqueness witnesses a **uniform strenghtening** of an original uniqueness statement.

A modulus of uniqueness always exists (computably) if X is (effectively) compact and T is (computable) continuous and has at most one fixed point and can be proof-theoretically extracted from a given (even noneffective) uniqueness proof.

Examples in best Chebycheff approximation (K. 1990, APAL 1992, Numer. Funct. Anal. Opt. 1993) and best L^1-approximation (K./Oliva, APAL 2003) from WKL-based uniqueness proofs.

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This found many applications in fixed point theory: Ariza-Ruiz, Briseid, Gerhardy, Jimenez-Melado, K., Lopéz-Acedo, Oliva.

Moduli of metric regularity for mappings

In many areas of analysis, in particular, continuous optimization notions of **linear** or **Hölder metric regularity** and **weak sharp minima** etc. play an important role which can be viewed as (often local forms of) special cases of (see also R.M. Anderson: 'Almost' implies 'Near', TAMS 1986) :

Definition

(X, d) metric space, $T : X \to X$ s.t. $F := Fix(T) \neq \emptyset$. T is **uniformly regular** with **modulus of metric regularity** $\phi : (0, \infty) \times (0, \infty) \to (0, \infty)$ for F if for all $d, \varepsilon > 0, x \in X$

$$d(x, F) \leq d \wedge d(x, Tx) < \phi(\varepsilon, d) \to \exists p \in F\, (d(x, p) < \varepsilon).$$

Usually this is only considered for linear moduli $\phi(d) \cdot \varepsilon$ or - recently - for Hölder-type moduli (Borwein/Li/Tam SIAM Optimiz. 2017).

What nonuniform property is this a uniformization of?

Answer: it is the uniformization of

$$\forall x \in X\,(Tx = x \to \forall \varepsilon > 0\,\exists p \in Fix(T)\,(d(x,p) < \varepsilon)),$$

which is **trivially true!**

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which is **trivially true!**

In fact, if X is **compact** and T is continuous a modulus of metric regularity always exists:

Proposition

If T is continuous, X is compact and $Fix(T) \neq \emptyset$, then T has a modulus of metric regularity.

Noncomputability of moduli of metric regularity

In general, there will be no computable moduli of metric regularity:

Proposition

There exists a computable firmly nonexpansive mapping $T : [0, 1] \to [0, 1]$ *which has no computable modulus of metric regularity* ϕ *w.r.t.* $Fix(T)$.

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The proof uses a construction due to E. Neumann LMCS 2015 of a computable firmly nonexpansive mapping $f : [0,1] \to [0,1]$ whose Picard iteration starting from 0 does not have a computable rate of convergence.

This can be recasted in terms of reverse mathematics:

Proposition

Over RCA_0, the statement that every continuous function $T : [0,1] \to [0,1]$ has modulus of metric regularity is equivalent to ACA.

Comment: The mere uniform metric regularity of continuous selfmaps of compact spaces is already provable in WKL_0.

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In fact, the cases where one can compute such a modulus are rare.

However there are important cases where this is true!

Examples of moduli of metric regularity

1. Let $T : X \to C \subseteq X$ be a retraction. Then $\phi(\varepsilon, d) := \varepsilon$ is a modulus of metric regularity since $Fix(T) = T(X) = C$ and $d(x, Tx) < \varepsilon$ implies that $dist(x, Fix(T)) < \varepsilon$ since $Tx \in Fix(T)$. In particular: if T is the metric projection of X onto C (when existent).

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2. Closed convex $C_1, C_2 \subseteq \mathbb{R}^n$: consider Douglas-Rachford operator

$$T_{C_1,C_2} := \frac{1}{2}(I + R_{C_2} R_{C_1}), \text{ where } R_{C_i} := 2P_{C_i} - I.$$

Borwein/Li/Tam SIAM 2017: if C_1, C_2 are convex semialgebraic sets in $\mathbb{R}^n$ with nonempty intersection which can be described by polynomials on $\mathbb{R}^n$ of degree d, then T_{C_1,C_2} has modulus of metric regularity (w.r.t. $Fix(T_{C_1,C_2})$)

$$\phi(\varepsilon, d) := (\varepsilon/\mu)^{-\gamma}$$

for all $x \in B_b(0)$ for suitable $\mu > 0$ and $\gamma \in (0, 1]$ depending on b, d.

Computational use of moduli of metric regularity

Definition

A sequence $\{x_n\}$ in a metric space (X, d) is Fejér monotone w.r.t. a subset $F \subseteq X$ if $\forall n \in \mathbb{N} \forall p \in F \; (d(x_{n+1}, p) \leq d(x_n, p))$.

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Proposition

Let $T : X \to X$ be with $Fix(T) \neq \emptyset$ and with modulus of metric regularity ϕ. Let $\{x_n\}$ be a sequence in X and $\psi : \mathbb{R}^*_+ \to \mathbb{N}$ be s.t.

$$\forall \varepsilon > 0 \, \exists n \leq \psi(\varepsilon) \ (d(x_n, Tx_n) < \varepsilon),$$

where $\{x_n\}$ is Fejér monotone w.r.t. $Fix(T)$. Then $\{x_n\}$ is Cauchy:

$$\forall \varepsilon > 0 \, \forall n, \tilde{n} \geq \Phi(\varepsilon) := \psi(\phi(\varepsilon/2)) \ (d(x_n, x_{\tilde{n}}) < \varepsilon)$$

and

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(2\varepsilon) \ (dist(x_n, Fix(T)) < \varepsilon).$$

If X is complete and T is continuous, then $\lim x_n \in Fix(T)$.

Metric regularity of families of sets and their intersection

Definition (Bauschke/Borwein 1996)

$C_1, \ldots, C_m, K \subseteq X$ with $C := \bigcap_{i=1}^m C_i \neq \emptyset$. $C_1, \ldots, C_m$ are **metrically regular w.r.t. K** with modulus ρ if

$$\forall \varepsilon > 0 \, \forall x \in K \, \left(\bigwedge_{i=1}^{m} dist(x, C_i) < \rho(\varepsilon) \rightarrow dist(x, C) < \varepsilon \right).$$

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Example (Borwein/Li/Yao SIAM Optimiz. 2014)

Let $C_1, \ldots, C_m \subseteq \mathbb{R}^n$ be basic convex semialgebraic sets given by

$$C_i := \{x \in \mathbb{R}^n | g_{i,j}(x) \leq 0, \, j = 1, \ldots m_i\},$$

where $g_{i,j}$ are convex polynomials on $\mathbb{R}^n$ with degree $\leq D \in \mathbb{N}$. Then for any compact $K \subseteq \mathbb{R}^n$ there exists $c > 0$ such that

$$\rho(\varepsilon) := (\varepsilon/c)^{-\gamma}/m, \; with \, \gamma := \left[\min \left\{ \frac{(2D-1)^n + 1}{2}, B(n-1)D^n \right\} \right]^{-1},$$

where $B(n) := \binom{n}{[n/2]}$, is modulus of metric regularity for $C_1, \ldots, C_m, K$.

Theorem

Let $T_1, \dots, T_m : X \to X$ *be with* $F := \bigcap_{i=1}^m Fix(T_i) \neq \emptyset$, $Fix(T_1), \dots, Fix(T_m)$ *metrically regular w.r.t.* K *with modulus* ρ. *For each* $i = 1, \dots, m$, *let* T_i *be metrically regular w.r.t.* $Fix(T_i)$ *with a common modulus* ϕ. *Let* $\{x_n\}$ *be a b-bounded Fejér (w.r.t.* F*) monotone sequence in* K *with*

$$\forall \varepsilon > 0 \, \exists n \leq \psi(\varepsilon) \, \left(\bigwedge_{i=1}^m d(x_n, T_i x_n) < \varepsilon \right).$$

Then $\{x_n\}$ *is a Cauchy sequence with Cauchy modulus*

$$\forall \varepsilon > 0 \, \forall n, \tilde{n} \geq \Phi(\varepsilon) := \psi(\phi(\rho(\varepsilon/2), b)) \, \left(d(x_n, x_{\tilde{n}}) < \varepsilon\right)$$

and

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(2\varepsilon) \, \left(dist(x_n, F) < \varepsilon\right).$$

If X *is complete,* F *is closed, then* $\lim x_n \in F$.

Corollary

Let $C_1, \ldots, C_m \subseteq X$ *be subsets of a metric space* (X, d) *with* $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$ *and* $T_i : X \to C_i$, $i = 1, \ldots, m$ *be retractions. Then under the assumptions on the metric regularity of* $C_1, \ldots, C_m$ *w.r.t.* K *and* $\{x_n\}$ *as in the theorem above one has* $\psi(\rho(\varepsilon/2))$ *as a Cauchy modulus for* $\{x_n\}$ *and - for complete* X *and closed* C *- the limit of* $\{x_n\}$ *belongs to* C.

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Let $C_1, \ldots, C_m \subseteq X$ be subsets of a metric space (X, d) with $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$ and $T_i : X \to C_i$, $i = 1, \ldots, m$ be retractions. Then under the assumptions on the metric regularity of $C_1, \ldots, C_m$ w.r.t. K and $\{x_n\}$ as in the theorem above one has $\psi(\rho(\varepsilon/2))$ as a Cauchy modulus for $\{x_n\}$ and - for complete X and closed C - the limit of $\{x_n\}$ belongs to C.

For a number of Fejér monotone iteration schemes $\{x_n\}$ rates of asymptotic regularity for common approximate fixed points have been established by proof-mining methods.

For metric projections T_i in Hilbert space H bounds for a schema due to Crombez are in **Khan/K. Nonlinear Analysis 2014**. Applied to $H := \mathbb{R}^n$ and **combined with Borwein/Li/Yao 2014** one gets:

Corollary

Let $P_i : \mathbb{R}^n \to C_i$ *be the metric projections onto convex semialgebraic sets* $C_1, \ldots, C_r \subseteq \mathbb{R}^n$ *with* $C := \bigcap_{i=1}^r C_i \neq \emptyset$ *as in the example. For* $1 \leq i \leq r$, *define* $T_i := Id + \lambda_i(P_i - Id)$ *for* $0 < \lambda_i \leq 2$, $\lambda_1 < 2$ *and put* $T := \sum_{i=1}^{r} \alpha_i T_i$, *where* $\alpha_1, \ldots, \alpha_r \in (0, 1)$ *with* $\sum \alpha_i = 1$. *Let* $x_0 \in \mathbb{R}^n$ *and* $D > \|x_0 - p\|$ *for some* $p \in C$ *and* $N_1, N_2 \in \mathbb{N}$ *s.t.*

$$\frac{1}{N_1} \leq \min\{\alpha_i \lambda_i : 1 \leq i \leq r\}, \ \frac{1}{N_2} \leq \min\{\alpha_1, 2 - \lambda_1\}.$$

Then for $x_n := T^n x_0$ *the conclusions of the theorem hold with*

$$\Phi(\varepsilon) := \left\lceil \frac{1936 \cdot N_1^6 \cdot (D+1)^4 (4N_1 + 1)^2 \cdot (2N_2 + 1)^2}{\pi \cdot \rho(\varepsilon/2)^4} \right\rceil,$$

where ρ *is the modulus from the example above.*

Metric regularity and minimization problems

Let X be a Hilbert space (or a CAT(0)-space) and $f : X \to (-\infty, +\infty]$ convex, lower semi-continuous and proper. Consider

$$\operatorname*{argmin}_{x\in X} f(x).$$

The **resolvent of f** of order $\lambda \in \mathbb{R}^*_+$:

$$J^f_\lambda(x) := \operatorname*{argmin}_{z\in X} \left[f(z) + \frac{1}{2\lambda} d(x,z)^2 \right],$$

J^f_λ is a firmly nonexpansive mapping and

$$Fix(J^f_\lambda) = \operatorname{argmin} f.$$

Definition (K./Lopéz-Acedo)

(X, d) metric space, $T_\lambda : X \to X$ family of maps for $\lambda \in I$ with $Fix(T_\lambda) = S \neq \emptyset$ for all $\lambda \in I$. ϕ is a modulus of metric regularity for S and $\{T_\lambda : \lambda \in I\}$ if

$$\forall d, \varepsilon > 0\, \exists \lambda \in I\, \forall x \in X$$
$$(d(x, S) \leq d \wedge d(x, T_\lambda x) < \phi(\varepsilon, d) \to \exists p \in S(d(x, p) < \varepsilon).$$

Quantitative analysis: Modulus of metric regularity for argmin f and modulus of uniform continuity of f can be converted into modulus for metric regularity for $\{J^f_{\lambda \in I} : \lambda \in I\}$ if I is unbounded.

In fact, one may take any $\lambda > d^2/\phi(\varepsilon, d)$ in the above definition (K./Lopéz-Acedo, 2015).

Recent success of proof mining in convex feasibility

Consider a Hilbert space H and nonempty closed and convex subsets $C_1, \ldots, C_N \subseteq H$ with metric projections P_{C_i}, define $T := P_{C_N} \circ \ldots \circ P_{C_1}$.
In 2003 Bauschke proved the 'minimal displacement conjecture':

$$\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).$$

Previously only known for $N = 2$ or $Fix(T) \neq \emptyset$ (or even $\bigcap_{i=1}^N C_i \neq \emptyset$) or C_i half spaces etc.

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K., February 2017: Proof mining extracts rate of convergence $\Phi(\varepsilon, N, b, K)$ which is a polynomial in ε, N and $b \geq \|x\|$ and $K \geq \|c := (c_1, \ldots, c_N)\|$ for some arbitrary $c \in C_1 \times \ldots \times C_N$.

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

with

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, \quad D := 2b + NK, \quad \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(4K^2 + N^3(N-1)^2 K^2)N^2}{\varepsilon}.$$