Randomness notions in Muchnik and Medvedev degrees

Kenshi Miyabe at Meiji University

Feb 24 2017

Dagstuhl Seminar on ”Computability Theory”

Background

Question

What is computational power with random access?

Can we construct a more random set from a given random set?

.

Old answer

Theorem 1 (De Leeuwe, Moore, Shannon, Shapiro (1956), Sacks)**.** *If $A \in 2^\omega$ is not computable, then the class*

$$\{X \in 2^\omega \ : \ A \leq_T X\}$$

has measure 0.

Thus, if A is computable with random access, then A is computable without random access.
How about the case there are many answers?

Mass problems

Definition 2. Let $P, Q \subseteq 2^\omega$.

P is Muchnik reducible to Q ($P \leq_w Q$) if, for every $f \in Q$, there exists $g \in P$ such that $g \leq_T f$.

P is Medvedev reducible to Q ($P \leq_s Q$) if, there exists a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.

The difference is uniformity.

Main results

Randomness hierarchy

$$\mathrm{KR} \supset \mathrm{SR} \supset \mathrm{CR} \supset \mathrm{MLR} \supset \mathrm{DiffR} \begin{matrix} \supset & \mathrm{W2R} & \supset \\ \supset & \mathrm{DemR} & \supset \end{matrix} 2\mathrm{R}$$

Muchnik degrees

$$\mathrm{KR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{matrix} <_w & \mathrm{W2R} & <_w \\ <_w & \mathrm{DemR} & <_w \end{matrix} 2\mathrm{R}$$

Medvedev degrees

$$\mathrm{SR} <_s \mathrm{CR}, \ \mathrm{MLR} <_s \mathrm{DiffR}$$

Proof

$\mathrm{KR} <_w \mathrm{SR}$

Let $\mathbf{a}$ be a minimal degree below $\mathbf{0}'$.
$\mathbf{a}$ is hyperimmune.
Every hyperimmune degree contains $X \in \mathrm{KR} \setminus \mathrm{SR}$.
Suppose $Y \leq_T X$ and $Y \in \mathrm{SR}$.
Since $\mathbf{a}$ is minimal, $Y \in \mathbf{a}$.
No minimal degree below $\mathbf{0}'$ can be high (Cooper '73), so Y is not high.
Nonhigh Schnorr random Y should be ML-random.
This contradicts to minimality of $\mathbf{a}$ by van Lambalgen's theorem.

MLR $<_s$ DiffR *(1/2)*

The goal is

$$\forall\Phi\exists X \in \mathrm{MLR}[\Phi(X) \notin \mathrm{DiffR}].$$

We can assume that Φ is almost-everywhere computable.
Let μ be the induced measure from Φ and the fair-coin measureλ, that is $\mu(U) = \lambda(\Phi^{-1}(U))$. Note that μ is computable.
If $\mu(\{Z\}) > 0$, then $\lambda(\Phi^{-1}(\{Z\})) > 0$. Hence, $\Phi^{-1}(\{Z\})$ contains a ML-random set X. However, every atom of a computable measure is computable.

$\mathrm{MLR} <_s \mathrm{DiffR}$ *(2/2)*

Levin-Kautz theorem says that, for a continuous measure ν and $\mathbf{a} > 0$, $\mathbf{a}$ contains ML-random iff $\mathbf{a}$ contains ν-ML-random. Apply this to μ and $\mathbf{0}'$ and get a μ-ML-random $Y \in \mathbf{0}'$.
By no-randomness-from-nothing for ML-randomness, there exists $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.

$\mathrm{SR} <_s \mathrm{CR}$

The goal is

$$\forall \Phi \exists X \in \mathrm{SR}[\Phi(X) \notin \mathrm{CR}].$$

When $\Phi = \mathrm{id}$, this means

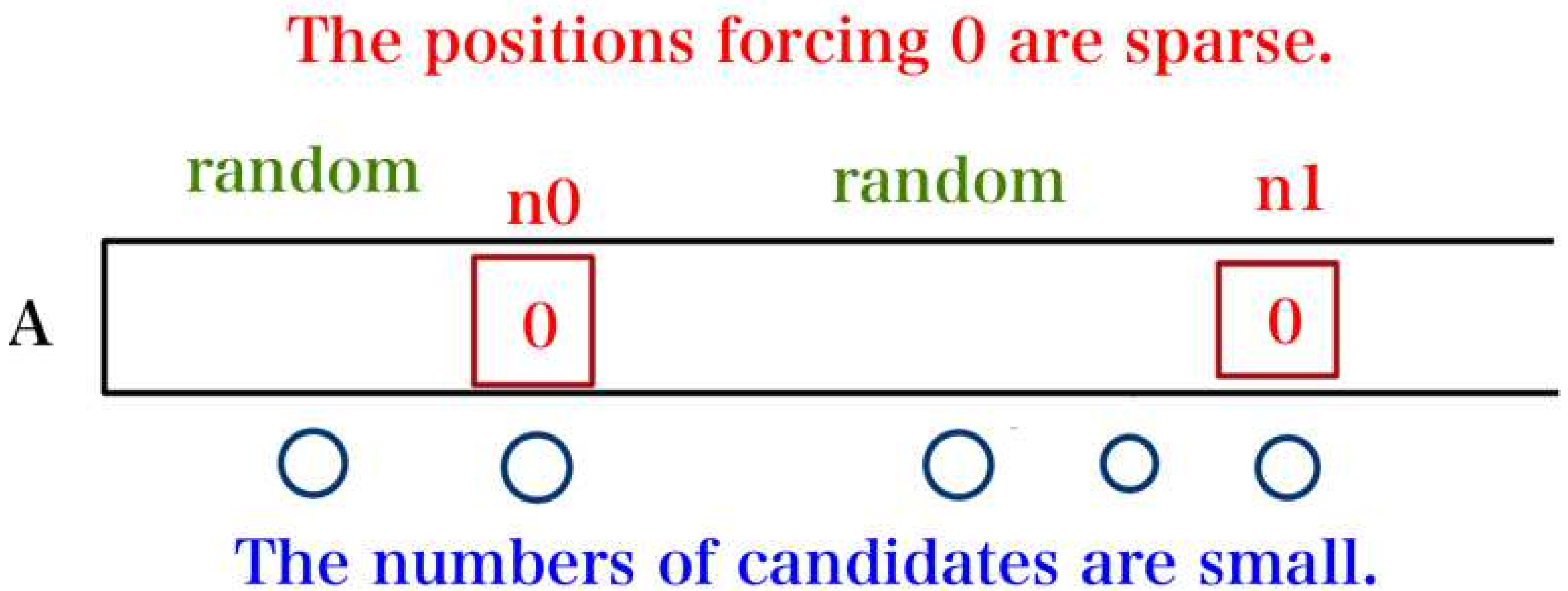
$$X \in \mathrm{SR} \setminus \mathrm{CR}.$$

Thus, we extend the method separating SR and CR.

id case

- Construct a random set A.
- Forcing $A(n_k) = 0$ in sparse positions
  $\Rightarrow$ too sparse not to be Schnorr random
- The number of candidates of n_k is small
  $\Rightarrow$ so small that some computable martingale succeeds on it.

$\mathrm{SR} \supsetneq \mathrm{CR}$



$\mathrm{SR} <_s \mathrm{CR}$

general case

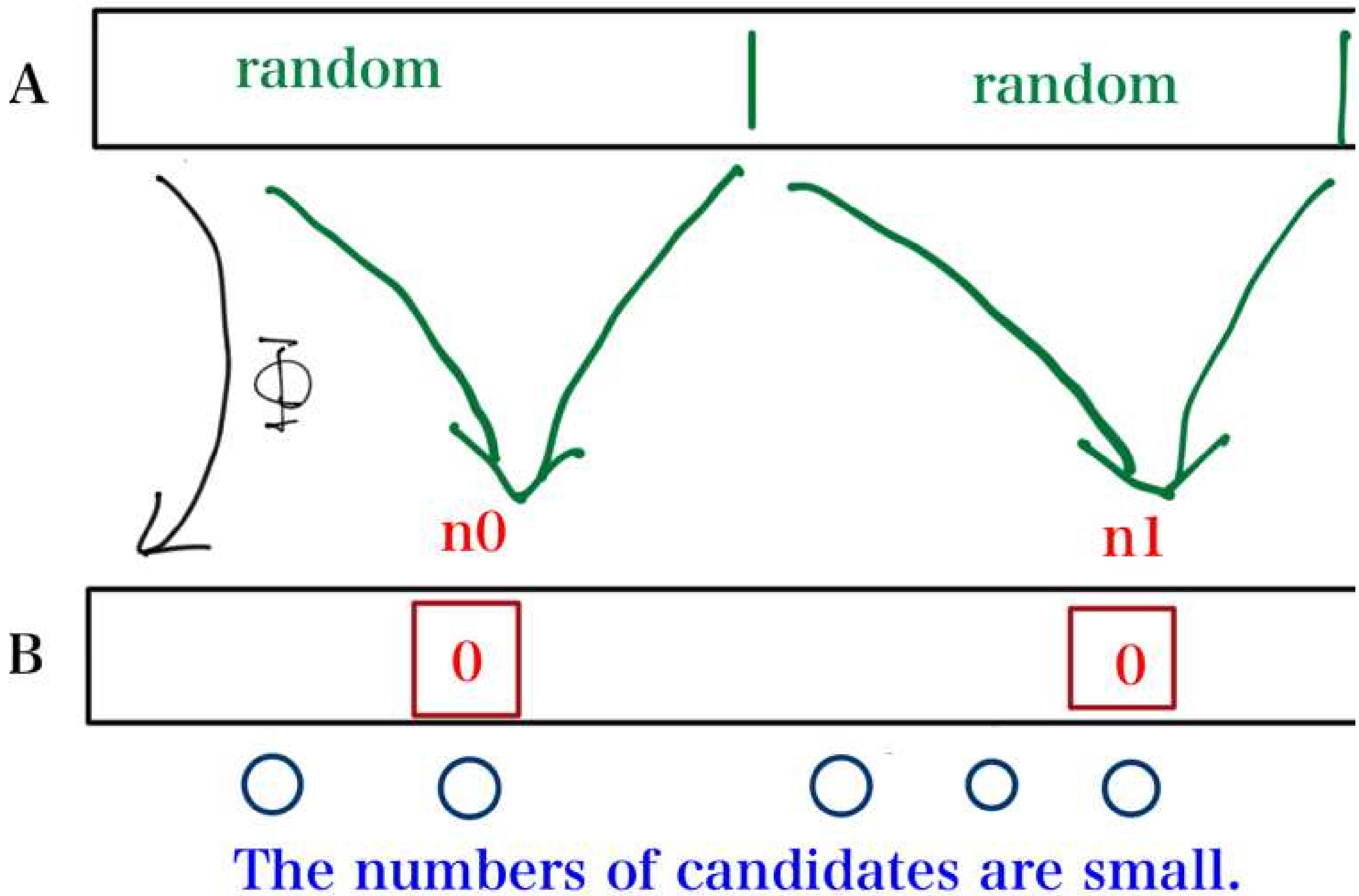
- Construct $A \in SR$ and $B = \Phi(A) \notin \mathrm{CR}$.
- Forcing $B(n_k) = 0$ in some positions (*)
- The number of candidates of n_k should be small $\Rightarrow B \notin \mathrm{CR}$.

The requirement (*) may be strong because

$$\lambda(\{X \in 2^\omega \ : \ \Phi(X)(n_k) = 0\})$$

may be too small (even empty).

$\mathrm{SR} <_s \mathrm{CR}$



$\mathrm{SR} <_s \mathrm{CR}$

We divide the case into two by the induced measure μ.

- μ is ”close to” uniform measure $(\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ the same method can be applied
- μ is ”far from” uniform measure $(\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ we can show it by a different reason

$\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

$$\exists Y \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$$

By no-randomness-from-nothing for CR,

$$\exists X \in \mathrm{CR}\, [\Phi(X) = Y].$$

Then, $X \in \mathrm{SR}$ and $\Phi(X) \notin \mathrm{CR}$.

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma 3 (essentially Bienvenu-Merkle). *Let μ, ν be computable measures.*

$$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\nu) \Rightarrow \mathrm{MLR}(\mu) \subseteq \mathrm{MLR}(\nu) \Rightarrow \mu \ll \nu$$

$\ll$ *means absolute continuity.*

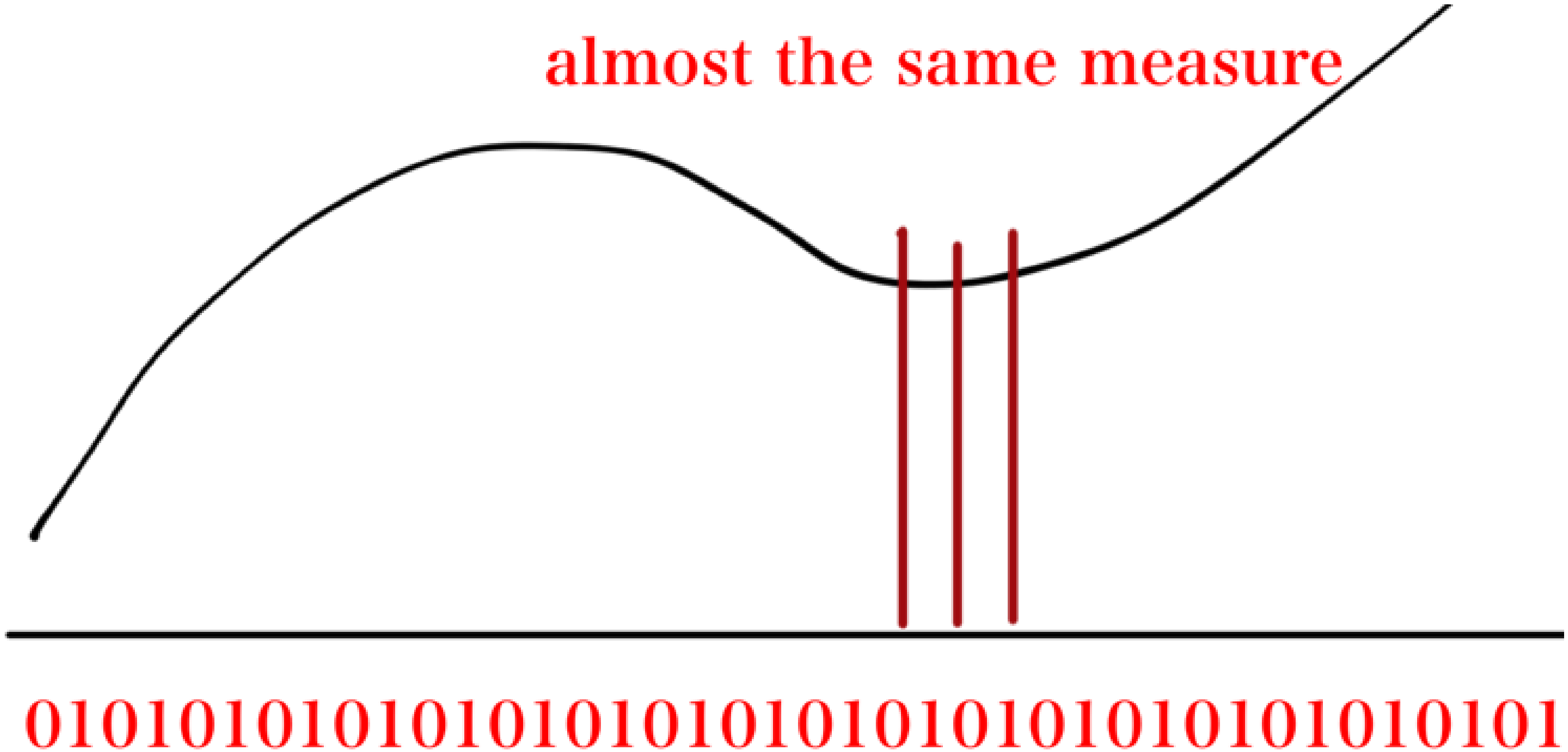
$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma 4. *Let Φ be an a.e. computable function. Let μ be the induced measure from Φ and λ. Assume $\lambda \ll \mu$. For each $\sigma \in 2^{<\omega}$,*

$$\lim_{n\to\infty} \lambda\{X \in [\sigma] \,:\, \Phi(X)(n) = 0\} = \frac{1}{2}\lambda(\sigma).$$

Proof. By the Radon-Nikodym theorem and Lévy's zero-one law. □

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$



Summary

Question

Question 5. Does there exist $X \in \mathrm{SR}$ such that, if $Y \leq_{tt} X$ then $Y \notin \mathrm{CR}$.
How about wtt?

I conjecture that we can not tt-compute (or wtt-compute) a computably random from a Schnorr random even nonuniformly.

Summary

- We found two problems that is possible non-uniformly but not possible uniformly.
- Analytical tools are useful to show results in computability. In particular, a.e. computable functions can be studied more from the measure-theoretic perspective.