A new result towards Fraïssé's conjecture conjecture.

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Dagstuhl, Germany

Reverse Mathematics

Reverse Mathematics refers to the program whose original motivating question is

"What set-existence axioms are necessary to do mathematics?"

asked in the setting of second-order arithmetic.

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- ACA_0: Arithmetic Comprehension $\iff$ "for every set X, X' exists".
- ATR_0: Arithmetic Transfinite recursion $\iff$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".
- Π^1_1-CA_0: Π^1_1-Comprehension $\iff$ "$\forall X$, the hyper-jump of X exists".

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Most of mathematics can be proved in Π^1_1-CA_0.

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FRA: The countable linear orderings form a WQO with respect to embeddablity.

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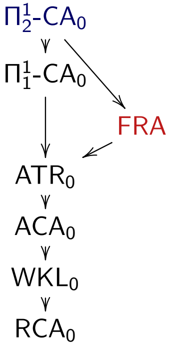

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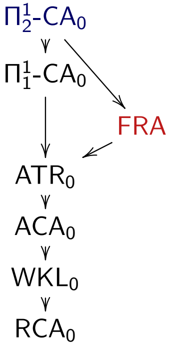
Obs: FRA $\not\Longrightarrow$ Π^1_1-CA_0. (No true Π^1_2 statement does.)



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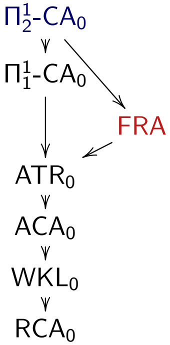
Theorem [Shore '93]: FRA $\Longrightarrow$ ATR_0 over RCA_0.



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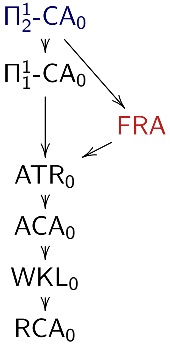

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Conjecture: [Clote '90][Simpson '99][Marcone]
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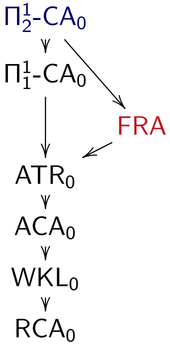

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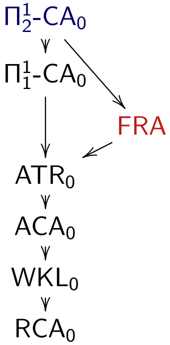

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Claim[M 05]: RCA_0+FRA is the least system where it is possible to develop a reasonable theory of embeddability of linear orderings.

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Theorem: FRA is equivalent to the following statements over RCA_0:

- [Kach,Marcone,M,Weiermann 11] For every ctble $\mathcal{L}$, there exists $n_{\mathcal{L}} \in \mathbb{N}$, such that: if $\mathcal{L}$ is colored with finitely many colors,

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- etc.

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Marcone and M. continued studying FRA in subsequent papers.

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[Marcone 96] A key lemma in Laver's proof, *the minimal bad array lemma*, implies Π^1_1-CA_0.

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Furthermore,
we prove FRA from a combinatorial statement weaker than Π^1_1-CA_0.

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Theorem [Simpson 85] (Π^1_1-TR) BQOs $\iff$ Borel BQOs.

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ATR_0+ 3 is Δ^0_2-bqo

|

FRA

|

ATR_0

Theorem [Simpson 85] (Π^1_1-TR) BQOs $\iff$ Borel BQOs.

Theorem: [M] (Π^1_1-CA_0) BQOs $\iff$ Δ^0_2-BQOs.

Theorem: [Marcone 05] (ATR_0) 3 is a BQO.

Corollary: (Π^1_1-CA_0) 3 is a Δ^0_2-BQO.

Obs: ATR_0+ 3 is Δ^0_2-BQO $\nvdash \Pi^1_1$-CA_0

Definition: Given $X \in \omega^\omega$, let X^- be X without its least element.

Definition: $(Q; \leq_Q)$ is a *Δ^0_2-Better-quasi-ordering (bqo)* if,
for every Δ^0_2 function $f : \omega^\omega \to Q$,
there is an $X \in \omega^\omega$ such that $f(X) \leq_Q f(X^-)$.

Theorem: [M.] ATR_0 + "3 is a Δ^0_2-BQO" implies FRA. • • •

Π^1_1-CA_0

$\Downarrow$

ATR_0+ 3 is Δ^0_2-bqo

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FRA

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ATR_0

Theorem [Simpson 85] (Π^1_1-TR) BQOs $\iff$ Borel BQOs.

Theorem: [M] (Π^1_1-CA_0) BQOs $\iff$ Δ^0_2-BQOs.

Theorem: [Marcone 05] (ATR_0) 3 is a BQO.

Corollary: (Π^1_1-CA_0) 3 is a Δ^0_2-BQO.

Obs: ATR_0+ 3 is Δ^0_2-BQO $\nvdash \Pi^1_1$-CA_0 because it's Π^1_2.