Weak truth table degrees of categoricity

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Motivating questions

- Study how computation interacts with various mathematical concepts.
- Complexity of constructions and objects we use in mathematics (how to calibrate?)
- Can formalize this more syntactically (reverse math, etc).
- Or more model theoretically...

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Motivating questions I: Presentation

- In computable model / structure theory, we place different effective (i.e. algorithmic) restrictions
  - presentations of a structure,
  - complexity of isomorphisms within an isomorphism type,
  - In this talk we want to focus on (Turing) degrees and interactions with these.
- For instance, classically, given any structure $\mathcal{A}$, a *copy* or a *presentation* is simply $\mathcal{B} = (\text{dom}(\mathcal{B}), R^{\mathcal{B}}, f^{\mathcal{B}}, \cdots)$ such that $\mathcal{B} \cong \mathcal{A}$.
- If $\mathcal{A}$ is countable and the language is computable, then this allows us to talk about $\deg(\mathcal{B})$.

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Motivating questions I: Presentation

- So one way of measuring precisely the complexity of a (non-computable) structure $\mathcal{A}$ might be to look at

$$\textit{Spec}(\mathcal{A}) = \{\deg(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}.$$

- This gives a finer analysis (of the classically indistinguishable).
- Extensive study of degree spectra.
- Difficulty: A countable $\mathcal{A}$ can have presentations of different Turing degrees, so it's not easy to define the "Turing degree" of a (class of) structures.

Motivating questions II: Complexity of Isomorphisms

- Let's look at another approach.
- Classically $\mathcal{A}$ and $\mathcal{B}$ are considered the same if $\mathcal{A} \cong \mathcal{B}$.
- However, from an effective point of view, even if $\mathcal{A} \cong \mathcal{B}$ are computable, they may have very different “hidden" effective properties.

Motivating questions II: Complexity of Isomorphisms

Example $(\omega, <)$

- Build a computable copy $\mathcal{A} \cong (\omega, <)$ where you arrange for $2n$ and $2n+2$ to be adjacent in $\mathcal{A}$ iff $n \notin \emptyset'$.
- "Successivity" was a hidden property that is made non-computable in some computable copy.

Example $(\omega, Succ)$

- This is rigid in a very uniform way.
- The entire structure is known once we fix 0.

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Motivating questions II: Complexity of Isomorphisms

- Since all definable properties are preserved by an isomorphism, it takes $\emptyset'$ to (Turing) compute an isomorphism between any two copies of $(\omega, <)$.
- However, (ω, Succ) is computably categorical.
- So $(\omega, <)$ appears to be more complicated than (ω, Succ), since accessing categoricity seems to require a more powerful oracle.
- This suggests another way of defining precisely the complexity of a structure:

Definition (Fokina, Kalimullin, Miller)

The degree of categoricity of a computable structure $\mathcal{A}$ is the least degree $\boldsymbol{d}$ such that $\boldsymbol{d}$ computes an isomorphism between any two computable copies of $\mathcal{A}$.

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Turing degrees of categoricity

- $(\omega, <)$ has degree of categoricity $\emptyset'$.
- $\emptyset$ is a degree of categoricity (for any c.c. structure).
- (Fokina, Kalimullin, Miller) Every d.c.e. degree (in and above $\emptyset^{(m)}$) is a degree of categoricity.
- (Csima, Franklin, Shore) Every d.c.e. degree (in and above $\emptyset^{(\alpha+1)}$) is a degree of categoricity.
- (Csima, N) Every Δ^0_2 degree is a degree of categoricity.
- (Csima, Franklin, Shore) All degrees of categoricity are hyperarithmetical.

Not Turing degrees of categoricity

- (Anderson, Csima) No 2-generic degree is a degree of categoricity.
- (Anderson, Csima) No hyperimmune-free degree is a degree of categoricity except $\emptyset$.
- (Anderson, Csima) Some Σ_2^0 degree is not a degree of categoricity.

Weak truth tables

Fact: Weak truth table (wtt) degrees are interesting/important.

So, we want to look at...

wtt-degrees of categoricity.

Definition

A weak truth table degree $\boldsymbol{a}$ is a wtt-degree of categoricity for a structure $\mathcal{A}$ if it is the least wtt-degree with the property that given any computable $\mathcal{A}_0 \cong \mathcal{A}_1 \cong \mathcal{A}$, there is an isomorphism $f : \mathcal{A}_0 \mapsto \mathcal{A}_1$ such that "f is wtt-reducible from $\boldsymbol{a}$".

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Weak truth tables

- What does "*f* is wtt-reducible from ***a***" mean? One possible interpretation is:

  *the output $f(n)$ can be computed from **a** with recursively bounded use.*

Proposition (Belanger, N)

Let $X \in 2^\omega$ be any set and and $\mathcal{A}$ be any computable equivalence structure or computable linear order. Then $\mathcal{A}$ is not X-categorical with respect to the above definition unless $\mathcal{A}$ is computably categorical.

- Likely true in many other natural classes.

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Fact

Every c.e. r-degree is a r-degree of categoricity (for a graph), where $r = btt, tt, wtt$.

Question

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Structures

- We investigate which natural (classes of) structures have wtt-degrees of categoricity:
  - Restrict to linear orders, equivalence structures.
- We find that (unsurprisingly?) *very few* structures have wtt-degrees of categoricity, in contrast to T-degrees of categoricity.
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Linear orders: let's investigate $(\omega, <)$

Lemma

*Let **a** be a wtt-degree. Then $(\omega, <)$ is **a**-categorical iff **a** $\geq_{wtt}$ D for each Δ^0_2 set D.*

Lemma

Given any Δ^0_2 set D there is a Δ^0_2 set A such that A $\not\leq_{wtt}$ D.

Theorem

$(\omega, <)$ has no wtt-degree of categoricity.

Proof.

Any set of high Turing degree relative to $\emptyset'$ can *wtt*-compute every Δ^0_2 set. Relativize the construction of a pair of high Turing degrees to $\emptyset'$. □

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Linear Orders

Question

Can we generalize to any computable well-ordering?

- The above example can be generalized to cover $\omega + \mathcal{L}$ for any Δ^0_2 categorical $\mathcal{L}$.

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Shuffle sums of finite linear orders do not have a wtt-degree of categoricity.

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Equivalence Structures

- Now let's look at computable equivalence structures.
- (Csima, N) The Turing degrees of categoricity for computable equivalence structures are exactly $deg_T(\emptyset)$, $deg_T(\emptyset')$ and $deg_T(\emptyset'')$ (what you expect).
- For wtt-degrees, the situation is less trivial.
- Trivial upperbounds:
  - Each computable equivalence structure E is $\emptyset''$-tt-categorical.
  - A computable equivalence structure is $\emptyset''$-m-categorical if and only if E is Δ_2^0-categorical.

Equivalence Structures

Theorem

The following classes of computable equivalence structures E do not have a non-zero wtt-degree of categoricity:

(i) *There is some $m \in \omega$ and some infinite limitwise monotonic set W such that for every $n \in W$, there are exactly m many E-classes of size n.*

(ii) *Every class in E has infinitely many E-classes of the same size.*

- The proof in each case is quite different.

Equivalence Structures

Theorem

Let E be a computable equivalence structure where all classes are finite. Suppose that

$$n \mapsto \text{number of } E\text{-classes of size } n,$$
$$x \mapsto \text{the least } n \text{ such that for every } m > n \text{ there are more than } x \text{ many } E\text{-classes of size } m,$$

are both total and computable.
Then E has wtt-degree of categoricity $deg_{wtt}(\emptyset')$.

Questions

- Characterize the $\emptyset'$-wtt-categorical structures.
- Find more examples of structures with wtt-degrees of categoricity.
- Thank you.

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