Characterising the oracles that know half of each computable set

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The Γ parameter of a Turing degree

For $Z \subseteq \mathbb{N}$ the lower density is defined to be

$$\underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0,n)|}{n}.$$

Recall that

$$\gamma(A) = \sup_{X \text{ computable}} \underline{\rho}(A \leftrightarrow X)$$

The Γ parameter was introduced by Andrews et al. (2013):

$$\Gamma(A) = \inf\{\gamma(Y) \colon\ Y \leq_T A\}.$$

Theorem (Monin, 2016, available on Logic Blog 2016)

$\Gamma(A)$ *is either* 0, *or* $1/2$, *or* 1. *Also* $\Gamma(A) = 0 \Leftrightarrow \exists f \leq_{\mathrm{T}} A$
$\forall g$ *computable, bounded by* $2^{(2^n)} \exists^\infty n \; f(n) = g(n)]$

Viewing $1 - \Gamma$ as a Hausdorff pseudodistance

For $Z \subseteq \mathbb{N}$ the upper density is defined by

$$\overline{\rho}(Z) = \limsup_n \frac{|Z \cap [0, n]|}{n}.$$

- For $X, Y \in 2^{\mathbb{N}}$ let $d(X, Y) = \overline{\rho}(X \triangle Y)$ be the upper density of the symmetric difference of X and Y
- this is a pseudodistance on Cantor space $2^{\mathbb{N}}$ (that is, two objects may have distance 0 without being equal).

Let $\mathcal{R} \subseteq \mathcal{A} \subseteq M$ for a pseudometric space (M, d). The Hausdorff distance is $d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S))$.

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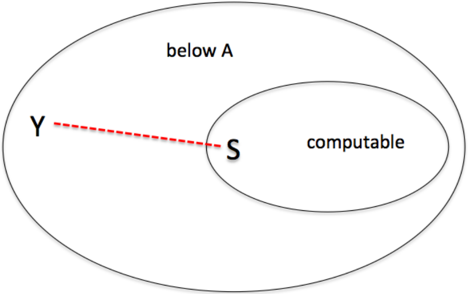
Given an oracle set A let $\mathcal{A} = \{Y : Y \leq_{\mathrm{T}} A\}$. Let $\mathcal{R} \subseteq \mathcal{A}$ denote the collection of computable sets. We have

$$1 - \Gamma(A) = d_H(\mathcal{A}, \mathcal{R}).$$



Δ parameter of a Turing degree

$$\begin{aligned}\delta(Y) &= \inf\{\underline{\rho}(Y \leftrightarrow S)\colon\ S \text{ computable}\} \\ \Delta(A) &= \sup\{\delta(Y)\colon\ Y \leq_T A\}.\end{aligned}$$

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- $\Gamma(A)$ measures how well computable sets can approximate the sets that A computes.
  "$\Gamma(A) > p$" for fixed $p \in [0, 1)$ is a lowness property.
- $\Delta(A)$ measures how well the sets that A computes can approximate the computable sets.
  "$\Delta(A) > p$" is a highness property.

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Properties of δ and Δ (w. Merkle and Stephan, Feb 2016)

- $\delta(Y) \leq 1/2$ for each Y (by considering also the complement of S)
- Y Schnorr random $\Rightarrow \delta(Y) = 1/2$ (by law of large numbers)

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- $\delta(Y) \leq 1/2$ for each Y (by considering also the complement of S)
- Y Schnorr random $\Rightarrow \delta(Y) = 1/2$ (by law of large numbers)
- A computable $\Rightarrow \Delta(A) = 0$.
- $\Delta(A) = 0$ is possible for noncomputable A, e.g. if A is low and c.e., or A is 2-generic.

The highness classes $\mathcal{B}(p)$

Definition (Brendle and N.)

For $p \in [0, 1/2)$ let

$$\mathcal{B}(p) = \{A\colon\ \exists Y \le_{\mathrm{T}} A\, \forall S \text{ computable } \underline{\rho}(Y \leftrightarrow S) > p\}.$$

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$$\Delta(A) > p \Rightarrow A \in \mathcal{B}(p) \Rightarrow \Delta(A) \geq p.$$

We will show that all the classes $\mathcal{B}(p)$ coincide, for $0 < p < 1/2$. Therefore:

$$\Delta(A) > 0 \Rightarrow \Delta(A) = 1/2.$$

Almost everywhere avoiding a comp. function

Definition ($\mathcal{B}(\neq^*, h)$, also known as $\mathtt{SNR}_h$)

For a function h, we let
$$\mathcal{B}(\neq^*, h) = \{A\colon \exists f \leq_{\mathrm{T}} A, f < h \,\forall g \text{ computable } \forall^\infty n\, f(n) \neq g(n)\}.$$

- This gets easier as h grows faster.
- The largest class $\mathcal{B}(\neq^*, \infty)$ coincides with "high or diagonally noncomputable".

  (Kjos-Hanssen, Merkle and Stephan, TAMS, Thm 5.1)
- outside the high sets, the hierarchy is closely related to the hierarchy of computing a DNR function below $\hat{h}$.

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Fact

A computes a Schnorr random $\Rightarrow$

$A \in \mathcal{B}(\neq^*, 2^{\hat{h}})$ whenever $\hat{h}$ is computable and $\infty > \sum_n 1/\hat{h}(n)$ is computable. E.g. $\hat{h}(n) = n^2$.

Main result

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Theorem (N., dual form of Monin's result)

$\mathcal{B}(p) = \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

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Recalling that $B(\neq^*, 2^{(2^n)}) \subseteq B(\neq^*, \infty) =$ high $\vee$ d.n.c:

Corollary

$\Delta(A) > 0 \Rightarrow$ A is high or d.n.c.

View as mass problems

We can also view $\mathcal{B}(p)$ and $\mathcal{B}(\neq^*, h)$ as mass problems (i.e. subsets of ω^ω). Re-define

$\mathcal{B}(p) = \{Y \in 2^{\mathbb{N}} \colon \forall S \text{ computable } \underline{\rho}(S \leftrightarrow Y) > p\}$.

$\mathcal{B}(\neq^*, h) = \{f < h \colon \forall g \text{ computable } \forall^\infty n\, g(n) \neq f(n)\}$.

Let $\leq_S$ denote uniform (or Medvedev) reducibility.
Unlike Monin's result, here we have Medvedev reductions.

Theorem (strengthens previous theorem)

$\mathcal{B}(p) \equiv_S \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

Easier direction (1)

Proposition

Let $p \in (0, 1/2)$. We have $\mathcal{B}(p) \geq_S \mathcal{B}(\neq^*, 2^{(2^n)})$.

Pick $a \in \mathbb{N}$ with $2/a < p$.

Claim (1)

$\mathcal{B}(2/a) \geq_S \mathcal{B}(\neq^*, 2^{(a^n)})$.

Proof.

Let (I_n) be the consecutive intervals in $\mathbb{N}^+$ of length a^n. Then $|I_n| > (a-1)|\bigcup_{k<n} I_k|$. So

$$X \in \mathcal{B}(2/a) \Rightarrow \forall U \text{ comp. } \forall^\infty n\, X \upharpoonright I_n \neq U \upharpoonright I_n$$

because $\underline{\rho}(X \leftrightarrow U^c) > 2/a$. The class on the right is Medvedev equivalent to $\mathcal{B}(\neq^*, 2^{(a^n)})$. □

Easier direction (2)

Proposition (recall)

Let $p \in (0, 1/2)$. We have $\mathcal{B}(p) \geq_S \mathcal{B}(\neq^*, 2^{(2^n)})$.

Claim (2)

$\mathcal{B}(\neq^*, h(n)) \equiv_S \mathcal{B}(\neq^*, h(2n))$ for each nondecreasing h.

Proof.

$\geq_S$ is trivial. For $\leq_S$:
Given $f \in \mathcal{B}(\neq^*, h(2n))$, let $g(2n) = g(2n+1) = f(n)$.
Then $g \in \mathcal{B}(\neq^*, h(n))$. □

Iterating this $\log_2 a$ times we get

$$B(\neq^*, 2^{(a^n)}) \equiv_S B(\neq^*, 2^{(2^n)}).$$

Sketch the harder direction $\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$:

Relation 1: Let $q > p$ such that $q < 1/2$. For $h(n) = 2^{\hat{h}(n)}$ and functions $x, y < h$, view $x(n)$ as string of length $\hat{h}(n)$.

$$x \neq^*_{\hat{h},q} y \Leftrightarrow \forall^\infty n \, |\{i < \hat{h}(n) \colon x(n)(i) \neq y(n)(i)\}| \geq \hat{h}(n)q.$$

Define $\mathcal{B}$-classes for these relations as before. Four steps:

1. there is k such that where $\hat{h}(n) = \lfloor 2^{n/k} \rfloor$
$$\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*_{\hat{h},q}).$$

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Relation 2: Let $L \in \mathbb{N}$ and u be a function. For a trace s consisting of L-element sets, and a function $y < u$, let

$$s \not\ni^*_{u,L} y \Leftrightarrow \forall^\infty n[s(n) \not\ni y(n)].$$

Define $\mathcal{B}$-classes for these relations as before. Four steps:

2. There are $L \in \mathbb{N}$, $\epsilon > 0$ such that where $u(n) = 2^{\lfloor \epsilon \hat{h}(n) \rfloor}$, we have $\mathcal{B}(\neq^*_{\hat{h},q}) \leq_S \mathcal{B}(\not\ni^*_{u,L})$ using error correction.

Sketch the harder direction $\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$:

Relation 2: Let $L \in \mathbb{N}$ and u be a function. For a trace s consisting of L-element sets, and a function $y < u$, let

$$s \not\ni^*_{u,L} y \Leftrightarrow \forall^\infty n[s(n) \not\ni y(n)].$$

Define $\mathcal{B}$-classes for these relations as before. Four steps:

3. $\mathcal{B}(\not\ni^*_{u,L}) \quad \leq_S \quad \mathcal{B}(\not\ni^*_{2^{(L2^n)},L})$.

Sketch the harder direction $\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$:

Relation 1: Let $q > p$ such that $q < 1/2$. For $h(n) = 2^{\hat{h}(n)}$ and functions $x, y < h$, view $x(n)$ as string of length $\hat{h}(n)$.

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Define $\mathcal{B}$-classes for these relations as before. Four steps:

1. there is k such that where $\hat{h}(n) = \lfloor 2^{n/k} \rfloor$
$$\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*_{\hat{h},q}).$$
2. There are $L \in \mathbb{N}$, $\epsilon > 0$ such that where $u(n) = 2^{\lfloor \epsilon \hat{h}(n) \rfloor}$, we have $\mathcal{B}(\neq^*_{\hat{h},q}) \leq_S \mathcal{B}(\not\ni^*_{u,L})$ using error correction.
3. $\mathcal{B}(\not\ni^*_{u,L}) \quad \leq_S \quad \mathcal{B}(\not\ni^*_{2^{(L2^n)},L})$.
4. Finally, $\mathcal{B}(\not\ni^*_{2^{(L2^n)},L}) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$

Separations?

By the easy direction above, $\mathcal{B}(0) \geq_S B(\neq^*, 2^{n!})$.

Question

Is $\mathcal{B}(1/4) \equiv_S \mathcal{B}(\neq^*, 2^{2^n}) >_W \mathcal{B}(0)$?

When do we know $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$? E.g.

- $g(n) = 2^{n^2}, h(n) = 2^{2^n}$, or
- $g(n) = 2^{2^n}, h(n) = 2^{n!}$?

Work in progress with Khan and Kjos-Hanssen, building on work of Khan and Miller on forcing with bushy trees:

- (Down) For each order function g there is order function h with $h > g$ such that $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$.
- (Up) For each order function h there is order function g with $h > g$ such that $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$.

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