## Weihrauch reducibility and recursion theory

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Weihrauch degrees and their structure

Classifying principles from recursion theory

Some open questions and speculations on computable model theory

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## Represented spaces and computability

## Definition

A *represented space* **X** is a pair  $(X, \delta_X)$  where *X* is a set and  $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$  a surjective partial function.

#### Definition

 $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$  is a realizer of  $f : \mathbf{X} \Rightarrow \mathbf{Y}$ , iff  $\delta_Y(F(p)) \in f(\delta_X(p))$  for all  $p \in \delta_X^{-1}(\operatorname{dom}(F))$ .



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#### Definition

 $f : \mathbf{X} \Rightarrow \mathbf{Y}$  is called computable (continuous), iff it has a computable (continuous) realizer.

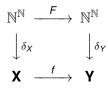
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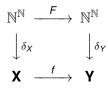
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## Weihrauch-reducibility

Definition For  $f :\subseteq \mathbf{X} \Rightarrow \mathbf{Y}, g :\subseteq \mathbf{V} \Rightarrow \mathbf{W}$  say

 $f \leq_W g$ 

iff there are computable  $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ , such that  $K \langle \operatorname{id}_{\mathbb{N}^{\mathbb{N}}}, GH \rangle$  is a realizer of f for every realizer G of g.

#### **Definition (Alternative)**

For  $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ , say that  $A \leq_W B$  if  $A = \emptyset$  or  $\exists n, m$  such that  $\forall x \in \mathbb{N}^{\mathbb{N}}$ , if  $\exists y \in \mathbb{N}^{\mathbb{N}} \langle x, y \rangle \in A$ , then

- 1.  $\Phi_n(x) \downarrow \text{ and } \exists y \in \mathbb{N}^{\mathbb{N}} \langle \Phi_n(x), y \rangle \in B$ )
- 2. If  $\langle \Phi_n(x), y \rangle$  for some  $y \in \mathbb{N}^{\mathbb{N}}$ , then  $\Phi_m \langle x, y \rangle \downarrow$  and  $\langle x, \Phi_m \langle x, y \rangle \rangle \in A$ .

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## What we know about structure

## Theorem (Brattka & Gherardi 2011; P. 2010) $\mathfrak{W}$ is a distributive lattice. The cartesian product $\times$ is an operation on $\mathfrak{W}$ .

#### Theorem (Higuchi & P. 2013)

 $\mathfrak{W}$  is not a Brouwer algebra.

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non-trivial countable suprema and only some non-trivial countable infima.

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Theorem (Higuchi & P. 2013)

 $\mathfrak{W}$  has no non-trivial countable suprema and only some non-trivial countable infima.

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## Embeddings into $\mathfrak{W}$

Theorem (Brattka & Gherardi 2011; Higuchi & P. 2013) For  $A \subseteq \mathbb{N}^{\mathbb{N}}$ , let  $d_A : A \to \{0\}$  and  $c_A : \{0\} \Rightarrow A$ . Then  $d_A : \mathfrak{M}^{op} \to \mathfrak{W}$  is a lattice embedding and  $c_A : \mathfrak{M} \to \mathfrak{W}$  is a meet-semilattice embedding.

#### Theorem

Let  $p, q \in \mathbb{N}^{\mathbb{N}}$  be Turing-incomparable. For  $A \subseteq \mathbb{N}$ , let  $e_A : \mathbb{N} \to \{p, q\}$  map  $n \in A$  to p and  $n \notin A$  to q. Then e. is an join-semilattice embedding of the many-one degrees into  $\mathfrak{W}$ .

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Definition (Brattka, Hendtlass & Kreuzer 2015) Call  $f :\subseteq \mathbb{N}^{\mathbb{N}} \Rightarrow \mathbb{N}^{\mathbb{N}}$  densely realized, if:

 $\forall p \in \mathbb{N}^{\mathbb{N}} \; \forall w \in \mathbb{N}^* \; \exists q \in \mathbb{N}^{\mathbb{N}} \; wq \in f(p)$ 

A Weihrauch degree is densely realized, if it has a densely realized representative.

Observation Problems from recursion theory are typically densely realized.

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## The separation

#### Proposition

# Let $f :\subseteq \mathbf{X} \Rightarrow \mathbb{N}$ and g be densely realized. If $f \leq_W g$ , then f is computable.

#### Definition

Let  $ACC_{\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$  be defined via  $n \in ACC_{\mathbb{N}}(p)$ , iff n + 1 is not the first non-zero entry in p.

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#### Observation

 $ACC_{\mathbb{N}}$  is reducible to every non-computable non-recursion theoretic theorem classified so far.

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## Some examples for connections

Theorem (Brattka & P. 2016) MLR  $\equiv_W C_{\mathbb{N}} \rightarrow WWKL.$ 

Theorem (Brattka, Hendtlass & Kreuzer 2015)

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1. COH  $\equiv_W$  lim  $\rightarrow$  WKL' 2. PA  $\equiv_W$  C'<sub>N</sub>  $\rightarrow$  WKL

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1. COH 
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## Further reading

#### V. Brattka, M. Hendtlass and A. Kreuzer. On the Uniform Computational Content of Computability Theory. arXiv, 1501.00433, 2015.

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## Uniform low basis theorem

## Definition Let $L :\subseteq \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be defined by q = L(p) iff $H^q = \lim p$ .

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# How does the local structure of the Weihrauch lattice look like? E.g.:

#### Question What can we say about the interval [WKL, lim]?



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Question What can we say about the interval [WKL, lim]? A uniform view on computable model theory I

#### Observation

Given some finite signature  $\mathcal{L}$ , there is a represented space  $\mathfrak{P}_{\mathcal{L}}$  of countable presentations of  $\mathcal{L}$ -structures.

#### Definition

Define  $\operatorname{Iso}_{\mathcal{L}} :\subseteq \mathfrak{P}_{\mathcal{L}} \times \mathfrak{P}_{\mathcal{L}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$  via  $(A, B) \in \operatorname{dom}(\operatorname{Iso}_{\mathcal{L}})$  iff  $A \cong B$ , and  $p \in \operatorname{Iso}_{\mathcal{L}}(A, B)$  if p is an  $\mathcal{L}$ -isomorphism from A to B. Let  $\operatorname{Iso}_{\mathcal{L}}^{A}$  be the restriction to presentations of A.

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## A uniform view on computable model theory II

Claim: The Weihrauch degree of  $Iso_{\mathcal{L}}^{\mathcal{A}}$  tells us (almost?) everything we want to know about the degree of categoricity of  $\mathcal{A}$ .

## Observation

Iso $_{\mathcal{L}}^{A} \equiv_{W}$  lim is the uniform version of saying that A has strong degree of categoricity  $\emptyset'$ . Iso $_{\mathcal{L}}^{A} \equiv_{W}$  lim should capture having degree of categoricity  $\emptyset'$ .

#### Conjecture

 $\operatorname{Iso}_{\mathcal{L}}^{\overline{A}} \equiv_{W} (\operatorname{Iso}_{\mathcal{L}}^{A})^{n}$  iff the spectral dimension of A is at most n.

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