Turing, tt-, and m-reductions for functions in the Baire hierarchy

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Computable reducibility for Type 2 functions

Motivating question: Suppose $f, g : [0,1] \to \mathbb{R}$.

What should $f \leq_T g$ mean?

When $f, g \in C([0,1])$, relative computability is defined in terms of representations of functions via sequences in 2^ω. For cardinality reasons, we cannot do that.

Yet we do have some intuitive idea that, for example

- A step function which steps at 0 should compute a step function that steps at 1/2.
- Equivalence classes should be closed under pointwise multiplication and addition of (continuous) computable functions.
- Given f, g, the degree of $f \oplus g$ should include the function

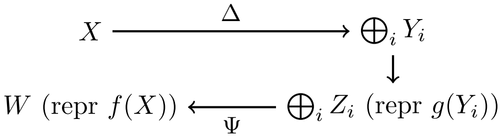
$$h(x) = \begin{cases} f(3x) & \text{if } x \leq 1/3 \\ g(3x-2) & \text{if } x \geq 2/3 \\ 0 & \text{otherwise.} \end{cases}$$

Parallelized Weihrauch reducibility

For technical convenience, consider now $f, g : 2^\omega \to \mathbb{R}$.

Definition 1. Say that $f \leq_T g$ if $f \leq_W \hat{g}$.

That is, $f \leq_T g$ if there are functionals Δ, Ψ such that, given $X \in 2^\omega$, the columns of $\Delta(X)$ are interpreted as an infinite sequence of inputs $\{Y_i\}$, and whenever $\{Z_i\}$ is a sequence of representations for $g(Y_i)$, then $\Psi(\bigoplus_i Z_i)$ is a representation for $f(X)$.



Parallelized Weihrauch reducibility

Definition 1. Say that $f \leq_T g$ if $f \leq_W \hat{g}$.

$$\begin{array}{ccc} X & \xrightarrow{\quad\Delta\quad} & \bigoplus_i Y_i \\ & & \downarrow \\ W \text{ (repr } f(X)) & \xleftarrow[\Psi]{\quad} & \bigoplus_i Z_i \text{ (repr } g(Y_i)) \end{array}$$

This satisfies the goals of the first slide, but:

Funny facts

- A step function which steps at a non-computable point does not compute a step function which steps at a computable point.
- There are continuous functions f, g such that f is reducible to g in the continuous degrees, but not under this reducibility.

Continuous Parallelized Weihrauch reducibility

Definition 2. Say that $f \leq_T g$ if there is some parameter $A \in 2^\omega$ relative to which $f \leq_W \hat{g}$.

$$\begin{array}{ccc} X & \xrightarrow{\Delta^A} & \bigoplus_i Y_i \\ & & \downarrow \\ W \text{ (repr } f(X)) & \xleftarrow[\Psi^A]{} & \bigoplus_i Z_i \text{ (repr } g(Y_i)) \end{array}$$

This "fixes" both funny facts, but it is a big coarsening.

Recall the Baire hierarchy of functions: $\mathcal{B}_0$ is the continuous functions and $\mathcal{B}_\alpha$ is the set of pointwise limits of functions from $\cup_{\beta<\alpha}\mathcal{B}_\beta$.

Proposition

- When restricted to $\cup_n \mathcal{B}_n$, the $\equiv_T$ classes are exactly the proper Baire classes $\mathcal{B}_{n+1} \setminus B_n$.
- This likely generalizes to all $\mathcal{B}_\alpha$.

Truth-table and many-one reducibility

The spirits of tt- and m-reducibility are:

- Truth-table: Say in advance exactly what bit of the oracle you will use, and what you will do with them.
- Many-one: Specify in advance exactly one bit of the oracle, and use its answer as your answer.

$$\begin{array}{ccc} X & \xrightarrow{\Delta^A} & \bigoplus_i Y_i \\ & & \downarrow \\ W \text{ (repr } f(X)) & \xleftarrow[\Psi^A]{} & \bigoplus_i Z_i \text{ (repr } g(Y_i)) \end{array}$$

Idea: Make Ψ^A a tt-reduction or an m-reduction.
(That is, A computes a truth table to apply to $\bigoplus_i Z_i$, or decides what entry of $\bigoplus_i Z_i$ to use for what entry of W.)

Cauchy name representation of a real doesn't make much sense for this.

Definition of tt- and m-reducibilities

Definition. We say $X \in 2^\omega$ is a *separation name* for $x \in \mathbb{R}$ if for all $p < q \in \mathbb{Q}$, we have

$$X(\langle p, q\rangle) = 0 \implies x < q \text{ and } X(\langle p, q\rangle) = 1 \implies x > p.$$

(So if $x \in (p, q)$, there is no restriction on $X(\langle p, q\rangle)$.)

Definition. We say $f \leq_{tt} g$ if there is some A relative to which $f \leq_W \hat{g}$, where the reverse computation is an A-computable tt-reduction.

Definition. We say $f \leq_m g$ if there is some A relative to which $f \leq_W \hat{g}$, where the reverse computation is an A-computable m-reduction.

Landmarks in the Baire hierarchy

Definition. Let $j_n : 2^\omega \to \mathbb{R}$ be defined by

$$j_n(X) = \sum_{i \in \omega} \frac{X^{(n)}(i)}{2^{i+1}}.$$

Fact. For each n, we have $j_n \in \mathcal{B}_n$.

Theorem. (Day, Downey, W.)

- For each n and f, if f is Baire but $f \notin \mathcal{B}_n$, then either

$$j_{n+1} \leq_m f \text{ or } -j_{n+1} \leq_m f.$$

- For each $f \in \mathcal{B}_n$, we have $f \leq_{tt} j_{n+1}$. (Probably holds for $\leq_m$ also.)

Proof: Uses $0^{(n)}$ priority argument.

Structure of Baire 1 functions

The Baire 1 functions support several ω_1-length ranking functions.

Consider the α, β and γ ranks studied by Kechris-Louveau (1990), corresponding to three different characterizations of the Baire 1 functions.

The α rank is defined as follows. Given $f \in \mathcal{B}_1$ and $p < q \in \mathbb{Q}$, let

- $P^0 = 2^\omega$,
- $P^{\nu+1} = P^\nu \setminus \cup\{U \text{ open } : f(U \cap P) \subseteq (p, \infty) \text{ or } f(U \cap P) \subseteq (-\infty, q)\}$
- $P^\nu = \cap_{\mu<\nu} P^\mu$ for ν a limit.

Let $\alpha(f, p, q)$ be the least α such that $P^\alpha = \emptyset$.
Let $\alpha(f) = \sup_{p<q\in\mathbb{Q}} \alpha(f, p, q)$.

The different ranks do not coincide generally, but:

Theorem. (Kechris, Louveau) If $f : 2^\omega \to \mathbb{R}$ is bounded, then for each ordinal ξ, we have $\alpha(f) \leq \omega^\xi$ iff $\beta(f) \leq \omega^\xi$ iff $\gamma(f) \leq \omega^\xi$.

Characterization of the $\leq_{tt}$ degrees in $\mathcal{B}_1$

For $f : 2^\omega \to \mathbb{R}$, let $\xi(f)$ be the least ξ such that $\alpha(f) \leq \omega^\xi$.

Theorem. (Day, Downey, W.) For $f, g \in \mathcal{B}_1$, we have $f \leq_{tt} g$ iff $\xi(f) \leq \xi(g)$.

Corollary. (Kechris-Louveau) If $f, g \in \mathcal{B}_1$ are bounded, then

$$\xi(f+g) \leq \max(\xi(f), \xi(g)).$$

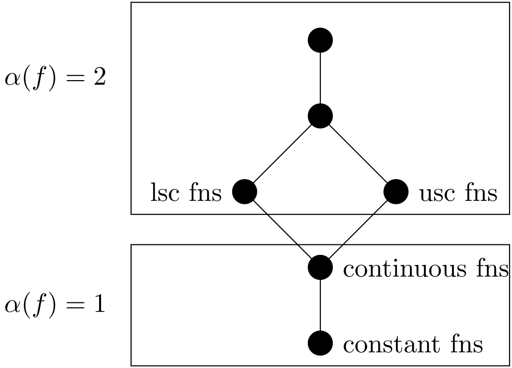
Proof: Observe that (using boundedness) $f + g \leq_{tt} f \oplus g$.

Characterization of the $\leq_m$-degrees in $\mathcal{B}_1$

Theorem. (Day, Downey, W.)

- If $\alpha(f) < \alpha(g)$, then $f <_m g$.
- If $\alpha(f) = \alpha(g)$ and this is a limit, then $f \equiv_m g$.
- If $\nu > 1$ is a successor, there are exactly 4 m-equivalence classes in $\{f : \alpha(f) = \nu\}$.

The initial segment of the m-degrees includes some recognizable classes.



Thank you.